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The Set of Dominance-Minimal Roots

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If α and β are positive roots in the root system of a Coxeter group W, we say that α dominates β if $w\beta$ is negative whenever $w\alpha$ is negative for $w \in W$. We say that α is elementary or dominance-minimal, if it does not dominate any $\beta \neq \alpha$. It is shown by the author and R. B. Howlett (*Math. Ann.* **296**, 1993, 179–190) that the set $\mathscr{E}$ of dominance-minimal roots is finite if and only if W has finite rank; this is used to show that W is automatic. To limit the size of the relevant automata, and possibly facilitate other Coxeter group algorithms, we give an explicit description of the set of elementary roots.

1. INTRODUCTION

The proof of Theorem (2.8) of [BH] yields for a Coxeter system (W, R) that $|\mathscr{E}|$ is bounded by

$$\sum_{d=1}^{c^{|R|}(|R|+1)+1} |R|^d = \frac{|R|}{|R|-1}\left(|R|^{c^{|R|}(|R|+1)+1} - 1\right),$$

where c equals the cardinality of the set

$$\{\cos(n\pi/m_{rs}) \mid r, s \in R, m_{rs} < \infty \text{ and } n \in \{1, \dots, m_{rs} - 1\}\}$$

(and m_{rs} denotes the order of rs). This bound, however, is rather large. For example, if

$$W = \langle r, s \mid r^2 = s^2 = (rs)^3 = 1\rangle,$$

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then $|\mathscr{E}| = 3$, but $|R| = 2$ and $c = 3$, and thus

$$\frac{|R|}{|R|-1}\left(|R|^{c^{|R|}(|R|+1)+1} - 1\right) = 2(2^{28} - 1).$$

Moreover, the algorithm given in [BH] to calculate the set of elementary roots is not very practical. In [C], Casselman asks for a more realistic algorithm and a better bound for $|\mathscr{E}|$. We will explicitly determine $\mathscr{E}$ for some Coxeter systems, and give an inductive formula for $\mathscr{E}$ for all other Coxeter systems; this will make it possible to calculate $|\mathscr{E}|$ precisely.

2. SUMMARY OF BACKGROUND MATERIAL

Let (W, R) be a Coxeter system as defined in [B, Hu]. For $r, s \in R$, let m_{rs} be the order of rs. Let $\Pi = \{a_r \mid r \in R\}$ be the set of *simple roots*, and for $a \in \Pi$ denote the element of R corresponding to a by r_a. Further, let V be the $\mathbb{R}$-vector space with basis Π, and let $\langle\ ,\ \rangle\colon V \times V \to \mathbb{R}$ be a symmetrical bilinear form which satisfies $\langle a_r, a_s\rangle = -\cos(\pi/m_{rs})$ for all $r, s \in R$ such that m_{rs} is finite, and $\langle a_r, a_s \rangle \leq -1$ if m_{rs} is infinite. Then $rv := v - 2\langle a_r, v\rangle a_r$ (for all $r \in R$ and $v \in V$) determines a faithful action of W on V which preserves $\langle\ ,\ \rangle$. The set $\Phi = \{wa_r \mid r \in R, w \in W\}$ is the *root system* of W on V.

For $S \subseteq \Pi$, the Coxeter group generated by $\{r_a \mid a \in S\}$ is denoted by W_S, and Φ_S denotes the root system of W_S on the subspace of V spanned by S. The *Coxeter graph* $\Gamma(S)$ of W_S has vertex set S, and two vertices a_r and a_s are adjoined by an edge or *bond* of *weight* m_{rs} if $m_{rs} \geq 3$ or $m_{rs} = \infty$; if $m_{rs} = 3$, the bond adjoining a_r and a_s is said to be *simple*, and if $m_{rs} \geq 4$ or $m_{rs} = \infty$, the bond adjoining a_r and a_s is labelled by m_{rs}.

Each root α can be written uniquely as $\sum_{x \in \Pi} \lambda_x x$; we say that λ_x is the *coefficient* $\operatorname{coeff}_x(\alpha)$ of x in α. The *support* $\operatorname{supp}(\alpha)$ of α is the set of all $x \in \Pi$ with $\operatorname{coeff}_x(\alpha) \neq 0$, and $\Gamma(\alpha) := \Gamma(\operatorname{supp}(\alpha))$ denotes the corresponding graph. It can be easily seen that $\Gamma(\alpha)$ is finite and connected. A root α is said to be *positive* (*negative*), if $\operatorname{coeff}_x(\alpha) \geq 0$ for all $x \in \Pi$ ($\operatorname{coeff}_x(\alpha) \leq 0$ for all $x \in \Pi$). The sets of positive and negative roots are denoted by Φ^+ and Φ^-, respectively. It is a well known fact that wa_r is positive for $w \in W$ and $r \in R$ if $l(wr) \geq l(w)$ (where the *length* $l(w)$ of w is the minimal integer n such that w equals $r_1 \cdots r_n$ for some $r_i \in R$). This yields that each root is either positive or negative. An extension of the proof of this fact gives the next proposition.

(2.1) PROPOSITION. *Let α be a root, $x \in \Pi$, and $\lambda = \operatorname{coeff}_x(\alpha)$.*

(i) *If $\lambda > 0$, then $\lambda \geq 1$.*

(ii) *If $\langle a, b\rangle \geq -1$ for all $a, b \in \Pi$, then λ is a polynomial in*

$$C = \left\{ \frac{\sin(l\pi/m)}{\sin(\pi/m)} \;\middle|\; 4 \leq m = m_{rs} < \infty \text{ for } a_r, a_s \in \Pi \text{ and } l \in \mathbb{N} \text{ with } l \leq \frac{m}{2} \right\}$$

with coefficients in $\mathbb{N}_0$ (the non-negative integers), or coefficients in $-\mathbb{N}_0$. In particular, if $0 < \lambda < 2$, then $\lambda = 1$ or $\lambda = 2\cos(\pi/m_{rs})$ for some $a_r, a_s \in \Pi$ with $4 \leq m_{rs} < \infty$.

Proof. Let $w \in W$ and $r \in R$ with $\alpha = wa_r$. If $l(w) = 0$, the assertion is trivially true; so suppose that $l(w) > 0$, and proceed by induction. Let $s \in R$ and $w' \in W$ such that $w = w's$ and $l(w') = l(w) - 1$. Next, choose $w'' \in wW_{\{a_r, a_s\}}$ of minimal length; then $l(w'') \leq l(ws) < l(w)$, and the inductive hypothesis applies to $w''a_r$ and $w''a_s$. Since $l(w''r)$, $l(w''s) \geq l(w'')$ by minimality of w'', the roots $w''a_r$ and $w''a_s$ are positive. Let $\mu_x, \nu_x \geq 0$ be the coefficients of x in $w''a_r$ and $w''a_s$, respectively. Further, let $u \in W_{\{a_r, a_s\}}$ with $w = w''u$; then $ua_r = \mu a_r + \nu a_s$ for some $\mu, \nu \in \mathbb{R}$, and $\lambda = \mu\mu_x + \nu\nu_x$.

If $\lambda > 0$, then $\mu, \mu_x > 0$ or $\nu, \nu_x > 0$. If $\mu > 0$, a straightforward calculation of the rank 2 case yields that $\mu \geq 1$ and $\nu \geq 0$; if $\mu_x > 0$, then $\mu_x \geq 1$ by the inductive hypothesis. Therefore $\lambda \geq 0 + \mu\mu_x \geq 1$ if $\mu, \mu_x > 0$; symmetrical arguments apply if $\nu, \nu_x > 0$, and this proves (i).

Suppose now that $\langle a, b\rangle \geq -1$ for all $a, b \in \Pi$, and denote the set of polynomials in C with coefficients in $\mathbb{N}_0$ by $\mathbb{N}_0[C]$. By the inductive hypothesis, and since $\mu_x, \nu_x \geq 0$, we know that $\mu_x, \nu_x \in \mathbb{N}_0[C]$. Further, either $\langle a_r, a_s\rangle = -1$, or $\langle a_r, a_s\rangle = -\cos(\pi/m_{rs})$ and $m_{rs} < \infty$; in both cases, it can be easily verified that μ and ν are both in $\mathbb{N}_0[C]$, or both in $-\mathbb{N}_0[C]$, and (ii) follows. ∎

It is well known that $\Phi_S = \Phi \cap V_S$ for $S \subseteq \Pi$, where V_S denotes the span of S in V; in particular, $\alpha \in \Phi_{\operatorname{supp}(\alpha)}$ for each root α. Therefore Proposition (2.1)(ii) can also be phrased as follows: If $\langle a, b\rangle \geq -1$ for all $a, b \in \operatorname{supp}(\alpha)$, then $\operatorname{coeff}_x(\alpha)$ is a polynomial in

$$C_\alpha := \left\{ \frac{\sin(l\pi/m)}{\sin(\pi/m)} \;\middle|\; 4 \leq m < \infty \text{ weight in } \Gamma(\alpha) \text{ and } l \in \mathbb{N} \text{ with } l \leq \frac{m}{2} \right\}$$

with coefficients in $\mathbb{N}_0$, or coefficients in $-\mathbb{N}_0$. In particular, if $0 < \operatorname{coeff}_x(\alpha) < 2$, then $\operatorname{coeff}_x(\alpha) = 1$ or $\operatorname{coeff}_x(\alpha) = 2\cos(\pi/m)$ for some weight m in $\Gamma(\alpha)$ with $4 \leq m < \infty$.

Following [S], we define for $\alpha \in \Phi^+$ the *depth* of α to be the minimal integer l such that $w\alpha$ is negative for some $w \in W$ of length l. It follows that $\alpha = ua_s$ for some $s \in R$ and $u \in W$ with $\mathrm{dp}(\alpha) = l(u) + 1$. In [BH], it is shown that for $r \in R$ and $\alpha \in \Phi^+ \backslash \{a_r\}$,

$$\mathrm{dp}(r\alpha) = \begin{cases} \mathrm{dp}(\alpha) - 1 & \text{if } \langle \alpha, a_r \rangle > 0, \\ \mathrm{dp}(\alpha) & \text{if } \langle \alpha, a_r \rangle = 0, \\ \mathrm{dp}(\alpha) + 1 & \text{if } \langle \alpha, a_r \rangle < 0. \end{cases} \tag{2.1}$$

For $\alpha, \beta \in \Phi^+$, we say that β *precedes* α (we write $\beta \preceq \alpha$) if and only if there exists a $w \in W$ such that $\alpha = w\beta$ with $\mathrm{dp}(\alpha) = l(w) + \mathrm{dp}(\beta)$; we write $\beta \prec \alpha$ if $\beta \preceq \alpha$ and $\beta \neq \alpha$. Note that it follows from Eq. (2.1) that for $r \in R$ and $\alpha \in \Phi^+ \backslash \{a_r\}$,

$$r\alpha \prec \alpha \Leftrightarrow \langle \alpha, a_r \rangle > 0.$$

It is clear that each positive root is preceded by a simple one. Furthermore, if $\beta \prec \alpha$, there exists an $x \in \mathrm{supp}(\alpha)$ with $\beta \preceq r_x \alpha \prec \alpha$: For if $\alpha = w\beta$ with $\mathrm{dp}(\alpha) = l(w) + \mathrm{dp}(\beta)$, let $u \in W$ and $x \in \Pi$ such that $w = r_x u$ and $l(w) = 1 + l(u)$; then $r_x \alpha = u\beta$ and

$$\mathrm{dp}(\alpha) - 1 \leq \mathrm{dp}(r_x \alpha) = \mathrm{dp}(u\beta) \leq l(u) + \mathrm{dp}(\beta) = \mathrm{dp}(\alpha) - 1,$$

hence $\beta \preceq r_x \alpha \prec \alpha$. In particular, if α is a non-simple positive root, there exists an $x \in \mathrm{supp}(\alpha)$ with $r_x \alpha \prec \alpha$. In [BH], it is shown that $\preceq$ defines a partial order on the set of positive roots. If $\beta \preceq \alpha$, an easy induction on $\mathrm{dp}(\alpha) - \mathrm{dp}(\beta)$ yields that the coefficients in β are less than or equal to the corresponding coefficients in α; moreover, if $\alpha = w\beta$ with $\mathrm{dp}(\alpha) = l(w) + \mathrm{dp}(\beta)$, then $w \in W_S$, where S is the set of all $x \in \Pi$ with $\mathrm{coeff}_x(\beta) < \mathrm{coeff}_x(\alpha)$.

For $\alpha, \beta \in \Phi^+$, we say that α *dominates* β with respect to W (we write $\alpha \operatorname{dom} \beta$) if and only if $w\beta \in \Phi^-$ whenever $w\alpha \in \Phi^-$ for $w \in W$. In [BH], it is shown that dom also defines a partial order on the set of positive roots. The dominance-minimal roots are called *elementary*, and the set of elementary roots is denoted by $\mathscr{E}$; that is, $\mathscr{E}$ is the set of positive roots that do not dominate any root other than themselves. From here on, when we refer to a minimal element of a set of (positive) roots, we mean precedence-minimal rather than dominance-minimal.

The next proposition summarizes some basic properties of dominance from [BH] (items (iv), (v), and (vi) are not stated explicitly, but follow easily from [BH, Lemma (2.3)]).

(2.2) PROPOSITION. *Let α and β be positive roots, $r \in R$, and $w \in W$. Then the following hold*:

(i) α dom β *if and only if* $\langle \alpha, \beta \rangle \geq 1$ *and* $\mathrm{dp}(\alpha) \geq \mathrm{dp}(\beta)$.

(ii) *If* α dom β *and* $w\beta \in \Phi^+$, *then* $w\alpha$ dom $w\beta$.

(iii) *If* $\beta \notin \mathscr{E}$ *and* $\beta \preceq \alpha$, *then* $\alpha \notin \mathscr{E}$.

(iv) *If* $\langle r\alpha, a_r \rangle \leq -1$ *and* $\alpha \neq a_r$, *then* $\alpha \notin \mathscr{E}$.

(v) *If* $\alpha \in \mathscr{E}$, *then* $r\alpha \in \mathscr{E}$ *if and only if* $\langle \alpha, a_r \rangle \in (-1, 1)$.

(vi) *If* $\beta \preceq \alpha$ *with* $\langle \beta, a_r \rangle \leq -1$ *and* $\mathrm{coeff}_{a_r}(\alpha) \neq \mathrm{coeff}_{a_r}(\beta)$, *then* $\alpha \notin \mathscr{E}$.

It follows readily from Proposition (2.2)(i) that a root α in Φ_S^+ is elementary in Φ_S^+ (with respect to W_S) if and only if it is elementary in Φ^+. This yields that $\Phi_S^+ \subseteq \mathscr{E}$ if W_S is finite: For in this case the subspace of V spanned by Φ_S is Euclidean, and so $\langle \alpha, \beta \rangle \in (-1, 1)$ for $\alpha, \beta \in \Phi_S^+$ with $\alpha \neq \beta$.

3. OUTLINE OF STRATEGY

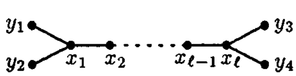
For $S \subseteq \Pi$, define $\mathscr{E}_S = \{\alpha \in \mathscr{E} \mid \mathrm{supp}(\alpha) = S\}$; then $\mathscr{E}$ is the disjoint union of all $\mathscr{E}_S$ with $S \subseteq \Pi$ such that $\Gamma(S)$ is finite and connected. If $S = \{x\}$, then $\mathscr{E}_S = \{x\}$. (In fact, $\mathscr{E}_S = \{\alpha \in \Phi^+ \mid \mathrm{supp}(\alpha) = S\}$ whenever W_S is finite.) Suppose next that $|S| > 1$, and assume that $\mathscr{E}_{S'}$ is already known for all proper subsets S' of S. In the subsequent sections, we shall determine $\mathscr{E}_S$ inductively, our strategy being as follows.

In Section 4, Lemma (4.1) yields that $\mathscr{E}_S = \varnothing$ if $\Gamma(S)$ is not a tree, or contains an infinite bond, and Proposition (4.7) gives an inductive formula for $\mathscr{E}_S$ whenever $\Gamma(S)$ is a tree which contains more than one non-simple bond. This leaves us to determine $\mathscr{E}_S$ for

(i) $S \subseteq \Pi$ with $\Gamma(S)$ a finite tree containing only simple bonds, and

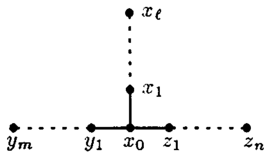
(ii) $S \subseteq \Pi$ with $\Gamma(S)$ a finite tree containing exactly one non-simple bond (of finite weight).

We proceed in Section 5 by determining $\mathscr{E}_S$ for S as in (i). Proposition (5.4) gives an inductive formula for $\mathscr{E}_S$ whenever $\Gamma(S)$ contains the subgraph



$(\ell \geq 1)$

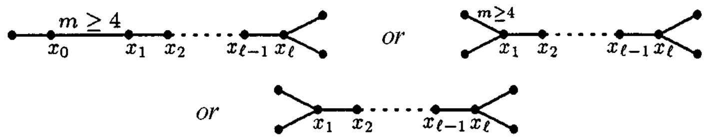
leaving us to determine $\mathscr{E}_S$ for $S \subseteq \Pi$ with $\Gamma(S)$ of shape



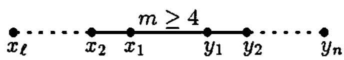
(0)

In Proposition (5.4), $\mathscr{E}_S$ is given explicitly for those S.

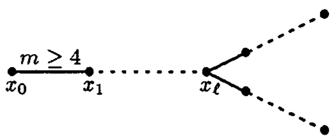
In Section 6, we deal with $\mathscr{E}_S$ for S satisfying (ii). Propositions (6.3), (6.4) give inductive formulas for $\mathscr{E}_S$ whenever $\Gamma(S)$ contains one of the following subgraphs:



(with $l \geq 1$). This leaves us to determine $\mathscr{E}_S$ for $S \subseteq \Pi$ with $\Gamma(S)$ of one of the following shapes:



(1)



(2)

If $\Gamma(S)$ is of shape (2), Proposition (6.6) gives a description of $\mathscr{E}_S$ in terms of $\mathscr{E}_T$ for T with $\Gamma(T)$ of shape (0) or (1). Since $\mathscr{E}_T$ for $\Gamma(T)$ of shape (0) is given in Proposition (5.4), this leaves us to determine $\mathscr{E}_T$ for $\Gamma(T)$ of shape (1); in Propositions (6.7), (6.8), and (6.12), $\mathscr{E}_T$ is listed explicitly for those T (for $m \geq 6$, $m = 4$, and $m = 5$, respectively).

4. INITIAL REDUCTION

The next lemma shows that $\mathscr{E}_S$ is empty unless $\Gamma(S)$ is finite, and does not contain any circuits or infinite bonds; that is, $\mathscr{E}_S = \varnothing$ unless $\Gamma(S)$ is a finite tree containing only finite weights.

(4.1) LEMMA. *Suppose that α is a positive root such that $\Gamma(\alpha)$ contains a circuit or an infinite bond. Then $\alpha \notin \mathscr{E}$.*

Proof. Assume that $\Gamma(\alpha)$ contains a circuit, and let $\beta \preceq \alpha$ be minimal such that $\Gamma(\beta)$ contains a circuit. We will show that $\beta \notin \mathscr{E}$; then $\alpha \notin \mathscr{E}$ by Proposition (2.2)(iii).

Let $r \in R$ with $r\beta \prec \beta$. Then $\Gamma(r\beta)$ does not contain any circuits by minimality of β. Hence a_r must be part of a circuit in $\Gamma(\beta)$ and $a_r \notin \operatorname{supp}(r\beta)$; that is, there exist at least two vertices b_1, b_2 of $\Gamma(\beta)$ adjoined to a_r, and $\operatorname{coeff}_{a_r}(r\beta) = 0$. Now $\langle a_r, b_i\rangle \leq -\cos(\frac{\pi}{3}) = -\frac{1}{2}$ for $i = 1, 2$ by definition of $\langle\ ,\ \rangle$, while $\langle a_r, x\rangle \leq 0$ for all other $x \in \Pi \setminus \{a_r\}$. Furthermore, $\operatorname{coeff}_{b_i}(r\beta) = \operatorname{coeff}_{b_i}(\beta) \geq 1$ for $i = 1, 2$ by Proposition (2.1)(i), and $\operatorname{coeff}_x(r\beta) \geq 0$ for all $x \in \Pi$. Therefore $\operatorname{coeff}_{b_i}(r\beta)\langle b_i, a_r\rangle \leq -\frac{1}{2}$ for $i = 1, 2$ and $\operatorname{coeff}_x(r\beta)\langle x, a_r\rangle \leq 0$ for $x \neq a_r$. Hence

$$\begin{aligned}\langle r\beta, a_r\rangle &= \operatorname{coeff}_{a_r}(r\beta)\langle a_r, a_r\rangle + \sum_{x \neq a_r} \operatorname{coeff}_x(r\beta)\langle x, a_r\rangle \\ &\leq 0 + \left(-\frac{1}{2}\right) + \left(-\frac{1}{2}\right) = -1,\end{aligned}$$

and thus $\beta \notin \mathscr{E}$ by Proposition (2.2)(iv), as required.

A similar argument applies when $\Gamma(\alpha)$ contains an infinite bond. ∎

The next assertion yields that a root α with coefficient 1 for some $x \in \Pi$ can be written as a "sum" (minus a correction factor) of roots, each of which has as support a subset of $\operatorname{supp}(\alpha)$; moreover, α is elementary if and only if each of the "summands" is elementary. This will enable us to determine $\mathscr{E}_S$ for "large" S inductively from $\mathscr{E}_{S'}$ for $S' \subset S$.

(4.2) PROPOSITION. *Let $S \subseteq \Pi$, $x \in S$, and $S_1, S_2 \subseteq S$ be such that $\Gamma(S_1 \setminus \{x\})$ and $\Gamma(S_2 \setminus \{x\})$ are unions of connected components of $\Gamma(S \setminus$*

$\{x\}$) with $S = S_1 \cup S_2$ and $S_1 \cap S_2 = \{x\}$. (Note that S_1 and S_2 uniquely determine one another.) Then

$$\begin{aligned}\phi\colon \big\{(\beta_1, \beta_2) \in \Phi_{S_1}^+ \times \Phi_{S_2}^+ \mid \operatorname{coeff}_x(\beta_1) = \operatorname{coeff}_x(\beta_2) = 1\big\} \\ \to \big\{\alpha \in \Phi_S^+ \mid \operatorname{coeff}_x(\alpha) = 1\big\} \\ (\beta_1, \beta_2) \mapsto \beta_1 + \beta_2 - x\end{aligned}$$

defines a one-one correspondence. Moreover, ϕ restricts to a one-one correspondence

$$\begin{aligned}\big\{(\beta_1, \beta_2) \in \mathscr{E}_{S_1} \times \mathscr{E}_{S_2} \mid \operatorname{coeff}_x(\beta_1) = \operatorname{coeff}_x(\beta_2) = 1\big\} \\ \leftrightarrow \big\{\beta \in \mathscr{E}_S \mid \operatorname{coeff}_x(\beta) = 1\big\}.\end{aligned}$$

Furthermore, $\operatorname{dp}(\phi(\beta_1, \beta_2)) = \operatorname{dp}(\beta_1) + \operatorname{dp}(\beta_2) - 1$ *and* $\beta_1, \beta_2 \preceq \phi(\beta_1, \beta_2)$ *for $\beta_i \in \Phi_{S_i}^+$ with* $\operatorname{coeff}_x(\beta_i) = 1$ $(i = 1, 2)$.

Note that an iteration of the above proposition yields the following "finer" decomposition of roots: Let $S_1, \dots, S_n \subseteq S$ be such that $S = S_1 \cup \cdots \cup S_n$ with $S_i \cap S_j = \{x\}$ and no vertex in $S_i \setminus \{x\}$ adjoined to any vertex in $S_j \setminus \{x\}$ for $i \neq j$ (for example, if the $\Gamma(S_i \setminus \{x\})$ are the connected components of $\Gamma(S \setminus \{x\})$. Then

$$(\beta_1, \dots, \beta_n) \mapsto \beta_1 + \cdots + \beta_n - (n-1)x$$

defines a one-one correspondence between the set of n-tuples in $\Phi_{S_1}^+ \times \cdots \times \Phi_{S_n}^+$ with coefficient 1 for x in each component, and the set of α in Φ_S^+ with $\operatorname{coeff}_x(\alpha) = 1$. Moreover, this map restricts to a one-one correspondence

$$\begin{aligned}\big\{(\beta_1, \dots, \beta_n) \in \mathscr{E}_{S_1} \times \cdots \times \mathscr{E}_{S_n} \mid \operatorname{coeff}_x(\beta_i) = 1 \text{ for } i = 1, \dots, n\big\} \\ \leftrightarrow \big\{\beta \in \mathscr{E}_S \mid \operatorname{coeff}_x(\beta) = 1\big\}.\end{aligned}$$

Before we prove Proposition (4.2), we establish two technical lemmas.

(4.3) LEMMA. *Let α and β be positive roots such that $\beta \preceq \alpha$. Further, let $y \in \Pi$ such that $y \neq \beta$ and* $\operatorname{coeff}_x(\beta) = \operatorname{coeff}_x(\alpha)$ *for all $x \in \Pi$ with $\langle x, y \rangle \neq 0$. Then $r_y \beta \preceq r_y \alpha$.*

Proof. Let $S = \{x \in \Pi \mid \operatorname{coeff}_x(\beta) < \operatorname{coeff}_x(\alpha)\}$; then there exists a $w \in W_S$ with $\alpha = w\beta$ and $\operatorname{dp}(\alpha) = l(w) + \operatorname{dp}(\beta)$. Since $\langle x, y \rangle = 0$ for $x \in S$, clearly $\langle \alpha, y \rangle = \langle \beta, y \rangle$ and $\tau_y w = w \tau_y$. Equation (2.1) yields that $\operatorname{dp}(r_y \alpha) - \operatorname{dp}(r_y \beta) = \operatorname{dp}(\alpha) - \operatorname{dp}(\beta) = l(w)$; furthermore,

$$r_y \alpha = r_y w \beta = (r_y w)\beta = (w r_y)\beta = w(r_y \beta).$$

Hence $r_y \beta \preceq r_y \alpha$. ∎

(4.4) LEMMA. *Let α be a positive root and $x \in \Pi$ with* $\operatorname{coeff}_x(\alpha) = 1$. *Then $x \preceq \alpha$; that is, there exists a $w \in W_{\operatorname{supp}(\alpha)\setminus\{x\}}$ such that $\alpha = wx$ and* $\operatorname{dp}(\alpha) = l(w) + 1$.

Proof. If α is of depth 1, the assertion is certainly true. Suppose now that $\operatorname{dp}(\alpha) > 1$, and proceed by induction. Let $y \in \Pi$ with $r_y \alpha \prec \alpha$; that is, $\langle \alpha, y \rangle > 0$.

If $y \neq x$, then $x \preceq r_y \alpha$ by the inductive hypothesis, and thus $x \preceq \alpha$ by transitivity of $\preceq$.

Suppose now that $y = x$. Then $\operatorname{coeff}_x(r_x \alpha) < 1$, and thus $\operatorname{coeff}_x(r_x \alpha) = 0$ by Proposition (2.1)(i); that is, $0 = \operatorname{coeff}_x(\alpha) - 2\langle \alpha, x \rangle = 1 - 2\langle \alpha, x \rangle$. So $\langle \alpha, x \rangle = \frac{1}{2}$. On the other hand,

$$\langle \alpha, x \rangle = \sum_{z \in \operatorname{supp}(\alpha)} \operatorname{coeff}_z(\alpha) \langle z, x \rangle = 1 + \sum_{z \neq x} \operatorname{coeff}_z(\alpha) \langle z, x \rangle.$$

As $\operatorname{coeff}_z(\alpha) \geq 1$ for $z \in \operatorname{supp}(\alpha)$ by Proposition (2.1)(i), it follows that x is adjoined to exactly one vertex z_0 of $\Gamma(r_x \alpha)$; moreover, $\langle x, z_0 \rangle = -\frac{1}{2}$ and $\operatorname{coeff}_{z_0}(r_x \alpha) = 1$. Therefore $z_0 \preceq r_x \alpha$ by the inductive hypothesis, and thus $r_x z_0 \preceq \alpha$ by Lemma (4.3). Since $r_x z_0 = x + z_0$ is certainly preceded by x, transitivity of $\preceq$ yields that $x \preceq \alpha$, as required. ∎

Proof of (4.2). It can be easily verified using Lemma (4.4) that ϕ is well defined and onto. Since ϕ is certainly one-one, it follows that ϕ is a one-one correspondence. Moreover, a straightforward induction on $l(w_1) + l(w_2)$ shows that $\phi(w_1 x, w_2 x) = (w_1 w_2)x$ and $\operatorname{dp}(w_1 w_2 x) = \operatorname{dp}(w_1 x) + \operatorname{dp}(w_2 x) - 1$ for $w_1 \in W_{S_1\setminus\{x\}}$ and $w_2 \in W_{S_2\setminus\{x\}}$; hence also $w_1 x, w_2 x \preceq \phi(w_1 x, w_2 x)$.

It remains to show for $w_i \in W_{S_i\setminus\{x\}}$ and $w := w_1 w_2$ that $wx \notin \mathscr{E}$ if and only if $w_1 x \notin \mathscr{E}$ or $w_2 x \notin \mathscr{E}$. Since $w_1 x$, $w_2 x \preceq wx$, Proposition (2.2)(iii) implies that $wx \notin \mathscr{E}$ if $w_1 x \notin \mathscr{E}$ or $w_2 x \notin \mathscr{E}$. For the converse, suppose that wx dominates $\gamma \in \Phi^+\setminus\{wx\}$. Then $w^{-1}\gamma \in \Phi^-$ by Proposition (2.2)(ii), since $x \in \mathscr{E}$. As $w \in W_{S\setminus\{x\}}$, in particular $x \notin \operatorname{supp}(\gamma)$. Since $\Gamma(\gamma)$ has to be connected, we may assume without loss of generality that $\gamma \in \Phi_{S_1\setminus\{x\}}$. Then $w_2^{-1}\gamma \in \Phi^+$, and therefore $w_1 x \operatorname{dom} w_2^{-1}\gamma$ by Proposition (2.2)(ii). Since $wx \neq \gamma$, clearly $w_1 x \neq w_2^{-1}\gamma$, and thus $w_1 x \notin \mathscr{E}$, as required. ∎

(4.5) COROLLARY. *Suppose that α is a root with $\Gamma(\alpha)$ a tree, and let $x, y \in \operatorname{supp}(\alpha)$ with* $\operatorname{coeff}_x(\alpha) = 1$. *Then* $\operatorname{coeff}_y(\alpha) \geq -2\langle x, y \rangle$. *Moreover, if x and y are adjoined by a simple bond (that is, $\langle x, y \rangle = -\frac{1}{2}$), then* $\operatorname{coeff}_y(\alpha) = 1$ *or* $\operatorname{coeff}_y(\alpha) \geq 2$.

Proof. As $x \preceq \alpha$ by Lemma (4.4), a straightforward induction on depth yields that $\operatorname{coeff}_y(\alpha) \geq -2\langle x, y \rangle$.

Now suppose that $\langle x, y\rangle = -\frac{1}{2}$. Let α be equal to $\beta_1 + \beta_2 - x$ according to Proposition (4.2), where $\Gamma(S_1 \setminus \{x\})$ is the connected component of $\Gamma(\operatorname{supp}(\alpha) \setminus \{x\})$ containing the vertex y and $\beta_1 \in \Phi^+_{S_1}$, $\beta_2 \in \Phi^+_{S_2}$. As $\Gamma(\alpha)$ is a tree and $\Gamma(S_1 \setminus \{x\})$ is connected, x is only adjoined to y in $\Gamma(S_1 \setminus \{x\})$; hence $\langle x, z\rangle = 0$ for $z \in S_1 \setminus \{x, y\}$. Therefore the coefficient of x in $r_x \beta_1$ equals

$$\begin{aligned}\operatorname{coeff}_x(\beta_1) - 2\langle x, \beta_1\rangle &= 1 - 2\sum_{z\in S_1} \operatorname{coeff}_z(\beta_1)\langle x, z\rangle \\ &= -1 - 2\operatorname{coeff}_y(\alpha)\langle x, y\rangle = \operatorname{coeff}_y(\alpha) - 1.\end{aligned}$$

By Proposition (2.1)(i), either $\operatorname{coeff}_x(r_x \beta_1) = 0$ or $\operatorname{coeff}_x(r_x \beta_1) \geq 1$, and the latter part of the assertion follows. ∎

(4.6) LEMMA. *Let α be a root such that $\Gamma(\alpha)$ contains the subgraph*

and $\operatorname{coeff}_{x_1}(\alpha), \ldots, \operatorname{coeff}_{x_l}(\alpha) > 1$. *Then* $\alpha \notin \mathscr{E}$.

Proof. If $\Gamma(\alpha)$ contains a circuit or an infinite bond, Lemma (4.1) yields the assertion. Suppose next that $\Gamma(\alpha)$ is a finite tree and contains only finite weights. As in the proof of Lemma (4.1), we let $\beta \preceq \alpha$ be minimal with coefficients greater than 1 for $x_1, \ldots, x_l$ and $x_0, x_{l+1} \in \operatorname{supp}(\beta)$. Denote the coefficient of $x \in \Pi$ in β by λ_x; then $\lambda_{x_i} \geq 2\cos(\frac{\pi}{4}) = \sqrt{2}$ for $i = 1, \ldots, l$ by Proposition (2.1)(ii).

Let $z \in \Pi$ with $r_z \beta \prec \beta$. Again, it suffices to show that $\langle r_z \beta, z\rangle \leq -1$; then β and all the roots preceded by β are non-elementary. Minimality of β forces $z = x_i$ for some $i \in \{0, \ldots, l+1\}$.

If $i = 0$, minimality of β yields further that $x_0 \notin \operatorname{supp}(r_{x_0} \beta)$, and thus

$$\langle r_{x_0} \beta, x_0\rangle = 0\langle x_0, x_0\rangle + \sum_{y \neq x_0} \lambda_y \langle y, x_0\rangle \leq \lambda_{x_1}\langle x_1, x_0\rangle.$$

Since $\lambda_{x_1} \geq \sqrt{2}$ and $\langle x_1, x_0\rangle = -\cos(\pi/m_1) \leq -\cos(\pi/4) = -\frac{1}{\sqrt{2}}$, this is less than or equal to -1, as required. Similar arguments apply if $i = l + 1$.

If $i \in \{1, \ldots, l\}$, then $\mathrm{coeff}_{x_i}(r_{x_i}\beta) \leq 1$ by minimality of β. Since $\Gamma(\beta)$ does not contain any circuits, and $\Gamma(r_{x_i}\beta)$ must be connected, Proposition (2.1)(i) forces $\mathrm{coeff}_{x_i}(r_{x_i}\beta) = 1$. If $i = 1$, then $\lambda_{x_0} \geq \sqrt{2}$ by Corollary (4.5) together with Proposition (2.1)(ii); hence

$$\lambda_{x_{i-1}}\langle x_{i-1}, x_i\rangle = \lambda_{x_0}\langle x_0, x_1\rangle \leq \sqrt{2}\left(-\cos(\pi/m_1)\right) \leq -1,$$

as $\cos(\pi/m_1) \geq \frac{1}{\sqrt{2}}$. If $i > 1$, Corollary (4.5) implies that $\lambda_{x_{i-1}} \geq 2$, and thus again $\lambda_{x_{i-1}}\langle x_{i-1}, x_i\rangle \leq -1$. Symmetrical arguments also yield that $\lambda_{x_{i+1}}\langle x_{i+1}, x_i\rangle \leq -1$, and therefore

$$\langle r_{x_i}\beta, x_i\rangle \leq 1 + \lambda_{x_{i-1}}\langle x_{i-1}, x_i\rangle + \lambda_{x_{i+1}}\langle x_{i+1}, x_i\rangle \leq -1,$$

as required. ∎

The following assertion is an immediate consequence of Proposition (4.2) together with the previous lemma.

(4.7) PROPOSITION. *Let $S \subseteq \Pi$ be such that $\Gamma(S)$ is a finite tree which contains the subgraph*

For $i \in \{1, \ldots, l\}$, let $\Gamma(S'_{i,1})$ be the connected component of $\Gamma(S \setminus \{x_i\})$ containing x_0, and set $S_{i,1} := S'_{i,1} \cup \{x\}$ and $S_{i,2} := S \setminus S'_{i,1}$. Then

$$\mathscr{E}_S = \bigcup_{i=1}^{l} \Big\{\alpha_i + \beta_i - x_i \mid \alpha_i \in \mathscr{E}_{S_{i,1}} \textit{ and } \beta_i \in \mathscr{E}_{S_{i,2}}$$
$$\textit{with } \mathrm{coeff}_{x_i}(\alpha_i) = \mathrm{coeff}_{x_i}(\beta_i) = 1\Big\}.$$

(*Note that the above union is not necessarily disjoint.*)

Observe that $S_{i,1}$ and $S_{i,2}$ defined above are proper subsets of S. Therefore $\mathscr{E}_{S_{i,1}}$ and $\mathscr{E}_{S_{i,2}}$ are already known by the inductive hypothesis, and Proposition (4.7) leaves us to determine $\mathscr{E}_S$ for $S \subseteq \Pi$ such that $\Gamma(S)$ contains at most one non-simple bond.

5. SIMPLE BONDS ONLY[1]

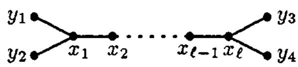
For the duration of this section, let $S \subseteq \Pi$ such that $\Gamma(S)$ is a finite tree which contains only simple bonds; we shall determine $\mathscr{E}_S$.

[1]A description of the set of elementary roots in this case has also been obtained by J.-Y. Hée in the unpublished article [He]. I am deeply indebted to Professor Hée for permitting many aspects of his treatment to be incorporated in the current paper.

Note that for $\alpha \in \Phi_S$, all the coefficients in α are integers, and $\langle \alpha, x \rangle$ is an integer multiple of $\frac{1}{2}$ for $x \in S$. In particular, if α is elementary and $\alpha \neq x$, then $\langle \alpha, x \rangle \leq \frac{1}{2}$; moreover, $r_x \alpha \prec \alpha$ if and only if $\langle \alpha, x \rangle = \frac{1}{2}$ in this case.

The statement of the next lemma is valid for all $S \subseteq \Pi$, including the ones for which $\Gamma(S)$ contains non-simple bonds. The proof of the general statement is slightly more involved, using tools which shall be introduced later. Therefore we will now merely give the proof for $S \subseteq \Pi$ with $\Gamma(S)$ a finite tree containing only simple bonds—using that $\operatorname{coeff}_x(\alpha) < 2$ implies that $\operatorname{coeff}_x(\alpha) \leq 1$ in this case. We shall indicate later how the proof can be generalized (compare the remark preceding the statement of Proposition (6.4)).

(5.1) LEMMA. *Suppose that α is a root such that $\Gamma(\alpha)$ contains only simple bonds, and contains the subgraph*



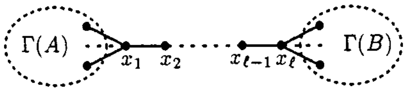
with $\operatorname{coeff}_{x_1}(\alpha), \dots, \operatorname{coeff}_{x_l}(\alpha) \geq 2$. *Then* $\alpha \notin \mathscr{E}$.

Proof. As in the proof of Lemma (4.1), let $\beta \preceq \alpha$ be minimal such that the coefficient λ_i of x_i in β is greater than or equal to 2 for all $i \in \{1, \dots, l\}$, and the coefficient μ_j of y_j in β is greater than 0 for all $j \in \{1, 2, 3, 4\}$. Using Proposition (2.2)(iii), (iv) again, it suffices to show that $\langle r_z \beta, z \rangle \leq -1$, where $z \in \operatorname{supp}(\beta)$ with $r_z \beta \prec \beta$.

Minimality of β forces $z = x_i$ for some $i \in \{1, \dots, l\}$, or $z = y_j$ for some $j \in \{1, 2, 3, 4\}$. If $z = y_j$, minimality of β yields further that $y_j \notin \operatorname{supp}(r_{y_j} \beta)$; since $\lambda_1, \lambda_l \geq 2$, we deduce that $\langle r_{y_j} \beta, y_j \rangle \leq -1$. If $z = x_i$, minimality of β yields that $\operatorname{coeff}_{x_i}(r_{x_i} \beta) < 2$; that is, $\operatorname{coeff}_{x_i}(r_{x_i} \beta) \leq 1$. Since $\mu_1, \mu_2, \mu_3, \mu_4 \geq 1$ by Proposition (2.1)(i), and $\lambda_j \geq 2$ for all j, it follows that $\langle r_{x_i} \beta, x_i \rangle \leq -1$, as required. ∎

The following assertion is an immediate consequence of Proposition (4.2) together with the previous lemma.

(5.2) PROPOSITION. *Let $S \subseteq \Pi$ be such that $\Gamma(S)$ is a tree which contains only simple bonds and equals*



where $A, B \subseteq S$ are disjoint sets with x_1 adjoined to at least two vertices of $\Gamma(A)$, and x_l adjoined to at least two vertices of $\Gamma(B)$, such that $\Gamma(A \cup \{x_1\})$ and $\Gamma(\{x_l\} \cup B)$ are connected. Then

$$\mathscr{E}_S = \bigcup_{i=1}^{l} \{\alpha_i + \beta_i - x_i \mid \alpha_i \in \mathscr{E}_{A \cup \{x_1, \ldots, x_i\}} \text{ and } \beta_i \in \mathscr{E}_{\{x_i, \ldots, x_l\} \cup B}$$
$$\text{have coefficient } 1 \text{ for } x_i\}.$$

(*Note that the above union is in general not disjoint.*)

Since $A \cup \{x_1, \ldots, x_i\}$ and $\{x_i, \ldots, x_l\} \cup B$ defined above are proper subsets of S, this leaves us to determine $\mathscr{E}_S$ for $S \subseteq \Pi$ with $\Gamma(S)$ of shape (0), as defined in Section 3.

For $S \subseteq \Pi$ with $\Gamma(S)$ of shape (0), we will give a root $\beta_{l,m,n} \in \mathscr{E}_S$ which precedes all elements of $\mathscr{E}_S$. So if $\alpha \in \mathscr{E}_S$, there exists a chain $\beta_{l,m,n} = \alpha_0 \prec \alpha_1 \prec \cdots \prec \alpha_d = \alpha$ of roots with $\mathrm{dp}(\alpha_i) = \mathrm{dp}(\alpha_{i-1}) + 1$ for all $i = 1, \ldots, d$. By Proposition (2.2)(iii), all α_i are elementary, and clearly $\mathrm{supp}(\alpha_i) = S$; therefore $\alpha_i \in \mathscr{E}_S$ for all i. Since $\alpha_{i+1} = r_i \alpha_i$ for some $r_i \in R$, and $\alpha_{i+1} \in \mathscr{E}$ if and only if $\langle \alpha_i, a_{r_i} \rangle \in (-1, 1)$ by Proposition (2.2)(iv), a straightforward induction on depth will thus enable us to enlist the elements of $\mathscr{E}_S$.

Define $\beta_{l,m,n} = ((r_{z_n} \cdots r_{z_2} r_{z_1})(r_{y_m} \cdots r_{y_2} r_{y_1})(r_{x_l} \cdots r_{x_2} r_{x_1}))x_0$. A straightforward induction on $l + m + n$, using Proposition (2.2)(v), yields that $\beta_{l,m,n}$ is elementary and equals

$$x_l + \cdots + x_2 + x_1 + y_m + \cdots + y_2 + y_1 + x_0 + z_1 + z_2 + \cdots + z_n.$$

The next lemma implies that every element of $\mathscr{E}_S$ is preceded by $\beta_{l,m,n}$.

(5.3) LEMMA. *Let α be an elementary root such that $\Gamma(\alpha)$ contains only simple bonds. Further, let β be a positive root with $\mathrm{coeff}_x(\beta) \leq \mathrm{coeff}_x(\alpha)$ for all $x \in \Pi$. Then $\beta \preceq \alpha$. In particular, β is also elementary.*

Proof. If $\mathrm{dp}(\alpha) = 1$, the assertion is certainly true. Suppose next that $\mathrm{dp}(\alpha) > 1$, and proceed by induction. Let $\alpha = \sum_{x \in S} \lambda_x x$ and $\beta =$

$\sum_{x \in S} \mu_x x$, and choose $y \in S$ with $r_y \alpha \prec \alpha$; then $\langle \alpha, y \rangle = \frac{1}{2}$ and $\mathrm{coeff}_y(r_y \alpha) = \lambda_y - 1$.

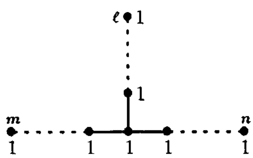
If $\mu_y \leq \lambda_y - 1$, the inductive hypothesis yields that $\beta \preceq r_y \alpha$, and the assertion follows by transitivity of $\preceq$. Next, suppose that $\mu_y > \lambda_y - 1$; that is, $\mu_y = \lambda_y$. If $\beta = y$, then $y \preceq \alpha$ by Lemma (4.4). It remains to consider the case that $\beta \neq y$. Since

$$\langle \beta, y \rangle = \sum_{x \in S} \mu_x \langle x, y \rangle = \mu_y \langle y, y \rangle + \sum_{x \neq y} \mu_x \langle x, y \rangle$$

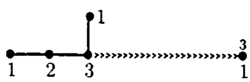
$$\geq \lambda_y \langle y, y \rangle + \sum_{x \neq y} \lambda_x \langle x, y \rangle = \langle \alpha, y \rangle = \frac{1}{2}, \qquad (*)$$

clearly $\mathrm{coeff}_y(r_y \beta) \leq \mu_y - 1 = \mathrm{coeff}_y(r_y \alpha)$. Therefore $r_y \beta \preceq r_y \alpha$ by the inductive hypothesis, and $r_y \beta \preceq \alpha$ by transitivity of $\preceq$. Since $\mathrm{coeff}_y(r_y \beta) \neq \mathrm{coeff}_y(\alpha)$ and $\alpha \in \mathscr{E}$, Proposition (2.2)(vi) yields that $\langle r_y \beta, y \rangle > -1$. On the other hand, $\langle r_y \beta, y \rangle = -\langle \beta, y \rangle \leq -\frac{1}{2}$ by $(*)$, and thus $\langle r_y \beta, y \rangle = -\frac{1}{2}$. We now have equality in $(*)$, and deduce that $\mu_x = \lambda_x$ for all $x \in \Pi$ with $\langle x, y \rangle \neq 0$; that is, $\mathrm{coeff}_x(r_y \beta) = \mathrm{coeff}_x(r_y \alpha)$ for all $x \in \Pi$ with $\langle x, y \rangle \neq 0$. Therefore $\beta = r_y(r_y \beta) \preceq r_y(r_y \alpha) = \alpha$ by Lemma (4.3). ∎

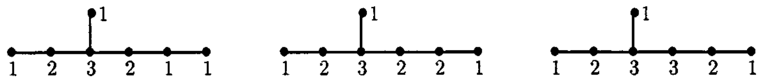
As outlined above, we are now able to use inductive arguments to enlist all elements of $\mathscr{E}_S$ for $S \subseteq \Pi$ with $\Gamma(S)$ of shape (0). We shall do so in Proposition (5.4); in order to state this proposition concisely, we need to introduce some more notation: Following [He], we identify $\beta_{l,m,n}$ with the diagram



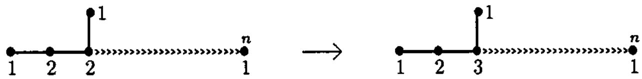
In another convenient notation also due to J.-Y. Hée, we write



as a shorthand notation for the roots

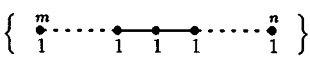


That is, the right hand branch has length 3, and the coefficients stay constant or decrease in steps of 1 alongside it. We write

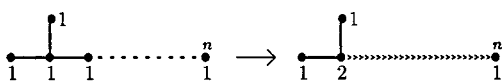


to indicate that each root on the right hand side is preceded by some root on the left hand side. Proposition (5.4) can now be phrased as follows.

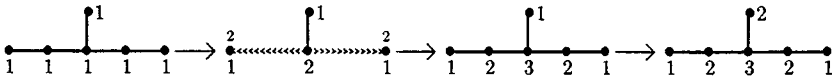
(5.4) PROPOSITION. *Let $S \subseteq \Pi$ be such that $\Gamma(S)$ is of shape* (0). *If $l = 0$, then*

$\mathcal{E}_S =$ 

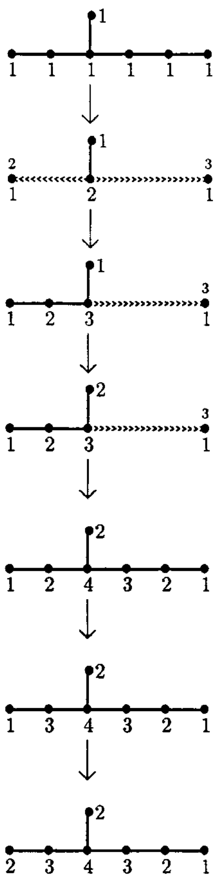
If $l = m = 1$ and $n \geq 1$, the elements of $\mathscr{E}_S$ are



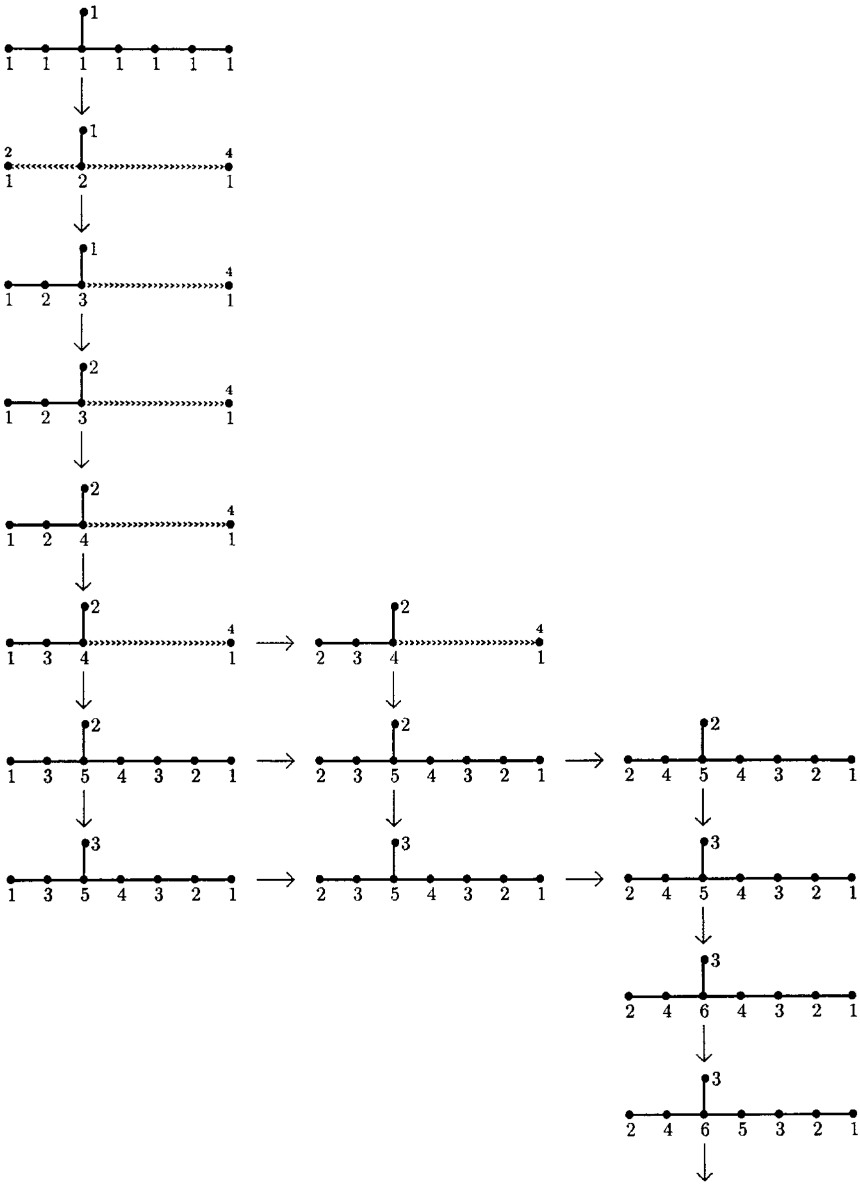
If $l = 1$, $m = 2$, and $n = 2$, the elements of $\mathscr{E}_S$ are

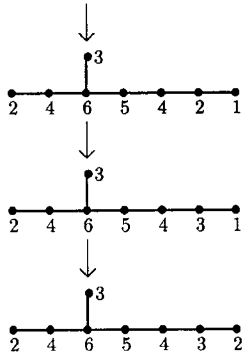


If $l = 1$, $m = 2$, and $n = 3$, the elements of $\mathscr{E}_S$ are

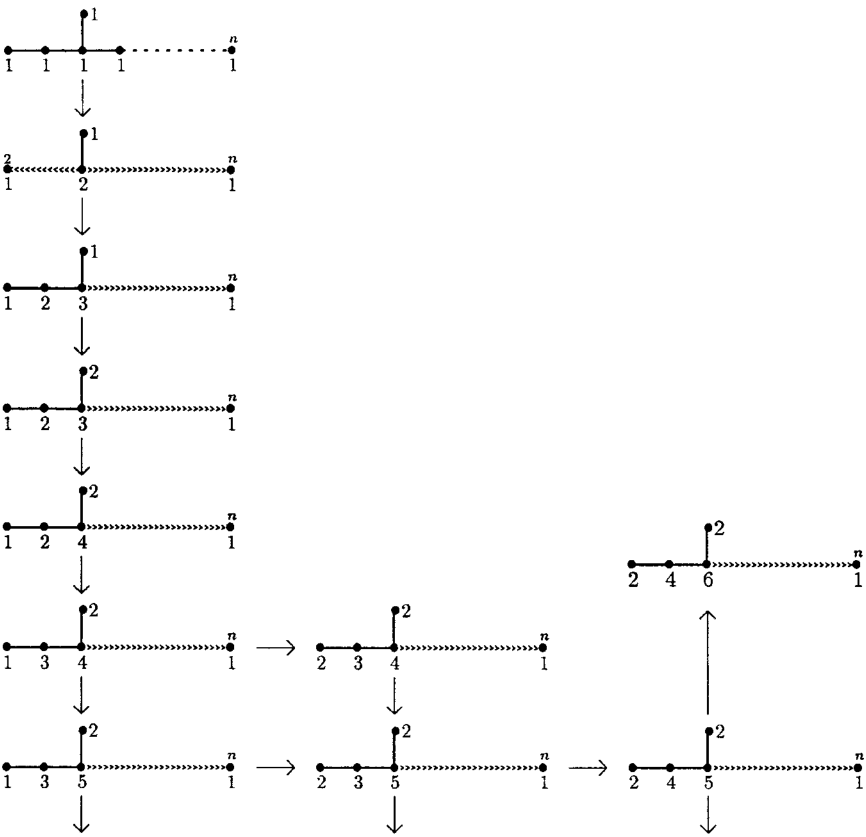


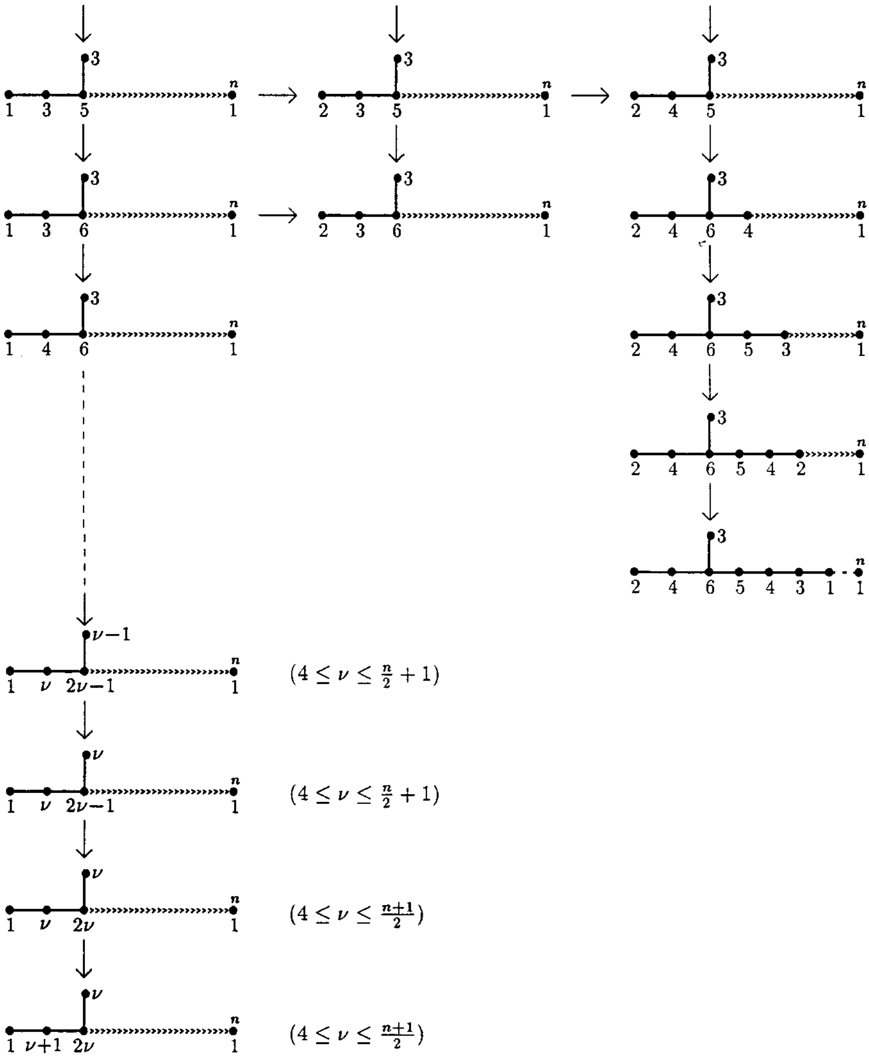
If $l = 1$, $m = 2$, *and* $n = 4$, *the elements of* $\mathscr{E}_S$ *are*





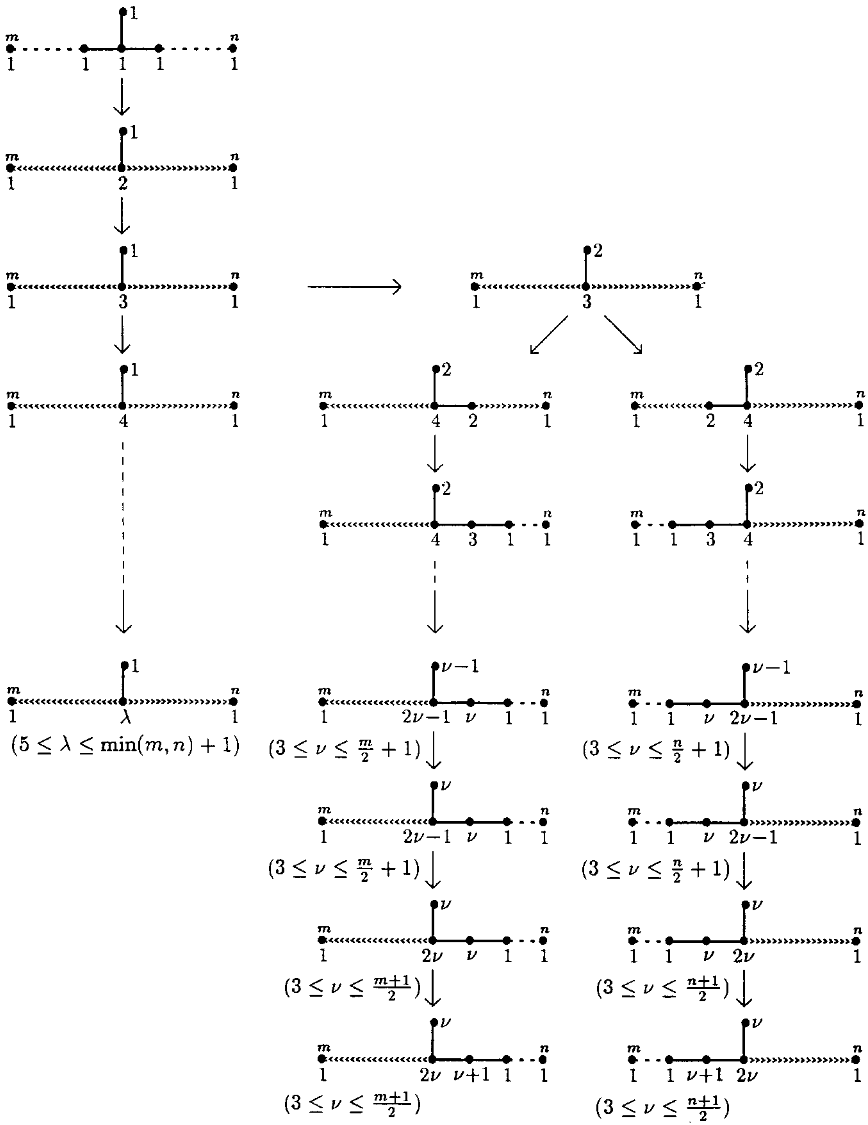
If $l = 1$, $m = 2$, and $n \geq 5$, the elements of $\mathscr{E}_S$ are



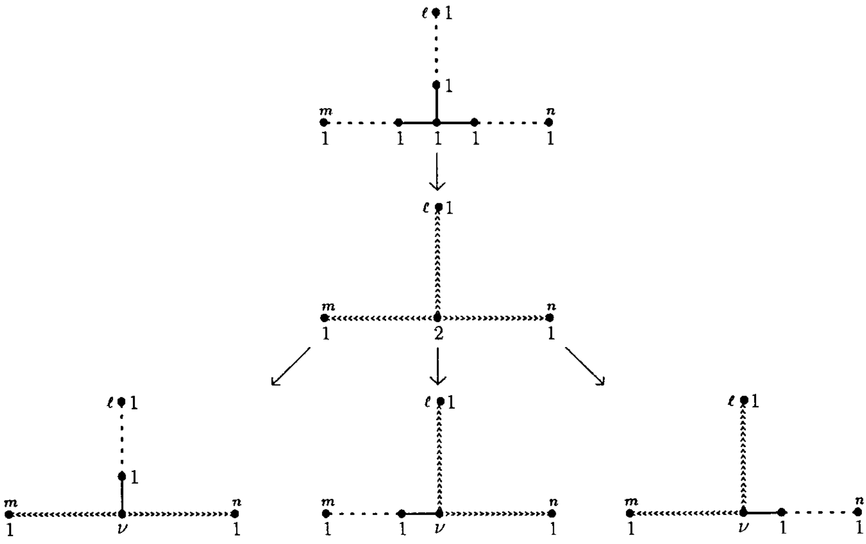


$(4 \leq \nu \leq \frac{n}{2} + 1)$
$(4 \leq \nu \leq \frac{n}{2} + 1)$
$(4 \leq \nu \leq \frac{n+1}{2})$
$(4 \leq \nu \leq \frac{n+1}{2})$

If $l = 1$ *and* $m, n \geq 3$, *the elements of* $\mathscr{E}_S$ *are*



If $l, m, n \geq 2$, the elements of $\mathscr{E}_S$ are



6. NON-SIMPLE BONDS

We will now determine $\mathscr{E}_S$ for $S \subseteq \Pi$ such that $\Gamma(S)$ contains a non-simple bond of finite weight $m \geq 4$. Set $c_m := 2\cos(\pi/m)$; then $r_y x = x + c_m y$ if $m_{r_x r_y} = m$ (that is, x and y are adjoined by a bond of weight m in S).

Similar to Proposition (4.2), the next assertion yields that a root α with coefficient c_m for some $x \in \Pi$ such that $\Gamma(\alpha)$ contains exactly one non-simple bond can be written as a "sum" (minus a correction factor) of roots, each of which has as support a subset of $\operatorname{supp}(\alpha)$; moreover, α is elementary if and only if each of the "summands" is elementary.

(6.1) PROPOSITION. *Let $S \subseteq \Pi$ be such that $\Gamma(S)$ is a finite tree which contains only bonds of weights 3 and m. Further, let x be in S, and suppose that $\Gamma(S_1')$ is a union of connected components of $\Gamma(S \setminus \{x\})$ such that*

$\Gamma(S_1' \cup \{x\})$ contains only simple bonds. Define $S_1 := S_1' \cup \{x\}$ and $S_0 := S \setminus S_1'$. Then

$$\phi\colon \left\{(\beta_0, \beta_1) \in \Phi_{S_0}^+ \times \Phi_{S_1}^+ \mid \operatorname{coeff}_x(\beta_0) = c_m, \operatorname{coeff}_x(\beta_1) = 1\right\}$$
$$\to \left\{\alpha \in \Phi_S^+ \mid \operatorname{coeff}_x(\alpha) = c_m\right\}$$
$$(\beta_0, \beta_1) \mapsto \beta_0 + c_m \beta_1 - c_m x$$

defines a one-one correspondence. Moreover, ϕ restricts to a one-one correspondence

$$\left\{(\beta_0, \beta_1) \in \mathscr{E}_{S_0} \times \mathscr{E}_{S_1} \mid \operatorname{coeff}_x(\beta_0) = c_m, \operatorname{coeff}_x(\beta_1) = 1\right\}$$
$$\leftrightarrow \left\{\alpha \in \mathscr{E}_S \mid \operatorname{coeff}_x(\alpha) = c_m\right\}.$$

Furthermore, $\operatorname{dp}(\phi(\beta_0, \beta_1)) = \operatorname{dp}(\beta_0) + \operatorname{dp}(\beta_1) - 1$ *and* $\beta_0 \preceq \phi(\beta_0, \beta_1)$ *for all* $\beta_0 \in \Phi_{S_0}^+$ *and* $\beta_1 \in \Phi_{S_1}^+$ *with* $\operatorname{coeff}_x(\beta_0) = c_m$ *and* $\operatorname{coeff}_x(\beta_1) = 1$.

Proposition (6.1) together with Proposition (4.2) yields that if $\Gamma(S_0 \setminus \{x\}), \dots, \Gamma(S_k \setminus \{x\})$ are pairwise disjoint unions of connected components of $\Gamma(S \setminus \{x\})$ such that $\Gamma(S_i)$ contains only simple bonds for $i \geq 1$, and $\Gamma(S_0)$ contains only bonds of weight 3 and m, then

$$(\beta_0, \dots, \beta_k) \mapsto \beta_0 + c_m(\beta_1 + \cdots + \beta_k - kx)$$

defines a one-one correspondence between the set of $(k+1)$-tuples in $\Phi_{S_0}^+ \times \cdots \times \Phi_{S_k}^+$ with coefficient c_m for x in the first component and coefficient 1 for x in all other components, and the set of roots in Φ_S^+ with coefficient c_m for x. Moreover, this map restricts to a one-one correspondence

$$\left\{(\beta_0, \dots, \beta_k) \in \mathscr{E}_{S_0} \times \cdots \times \mathscr{E}_{S_k} \mid \operatorname{coeff}_x(\beta_0) = c_m \text{ and } \operatorname{coeff}_x(\beta_i) = 1 \text{ for } i = 1, \dots, k\right\}$$
$$\leftrightarrow \left\{\alpha \in \mathscr{E}_S \mid \operatorname{coeff}_x(\alpha) = c_m\right\}.$$

Furthermore, $\operatorname{dp}(\phi(\beta_0, \dots, \beta_k)) = \operatorname{dp}(\beta_0) + \cdots + \operatorname{dp}(\beta_k) - k$ and $\beta_0 \preceq \phi(\beta_0, \dots, \beta_k)$ for all $\beta_0 \in \Phi_{S_0}^+$ with $\operatorname{coeff}_x(\beta_0) = c_m$ and $\beta_i \in \Phi_{S_i}^+$ with $\operatorname{coeff}_x(\beta_i) = 1$ for $i \geq 1$.

Proof of (6.1). We show first that ϕ is well defined. Let $(\beta_0, \beta_1) \in \Phi_{S_0}^+ \times \Phi_{S_1}^+$ with $\operatorname{coeff}_x(\beta_0) = c_m$ and $\operatorname{coeff}_x(\beta_1) = 1$. Lemma (4.4) implies that there exists a $w_1 \in W_{S_1'}$ of length $\operatorname{dp}(\beta_1) - 1$ with $\beta_1 = w_1 x$. Define $\alpha = w_1 \beta_0$. It can be easily seen that $\operatorname{coeff}_x(\alpha) = c_m$, and a straightforward induction on $l(w_1)$ yields that

$$\alpha = w_1 \beta_0 = \beta_0 + c_m(w_1 x - x) = \beta_0 + c_m(\beta_1 - x) = \phi(\beta_0, \beta_1) \quad (*)$$

and $\operatorname{dp}(\phi(\beta_0, \beta_1)) = \operatorname{dp}(\beta_0) + \operatorname{dp}(\beta_1) - 1$. Therefore ϕ is well defined. We can furthermore deduce that $\beta_0 \preceq \phi(\beta_0, \beta_1)$.

It is clear that ϕ is one-one, and we show now that ϕ is onto. Let $\alpha \in \Phi_S^+$ have coefficient c_m for x, and choose $\gamma \preceq \alpha$ minimal with $\operatorname{coeff}_x(\gamma) = c_m$.

Assume for a contradiction that x is adjoined to at least two vertices $y_1 \neq y_2$ in $\operatorname{supp}(\gamma)$. Then $\Gamma(\operatorname{supp}(\gamma) \setminus \{x\})$ is not connected. Therefore $\operatorname{coeff}_x(r_x\gamma) > 0$ by connectedness of $\Gamma(r_x\gamma)$. On the other hand, minimality of γ forces $r_x\gamma \prec \gamma$; that is, $\langle x, \gamma \rangle > 0$. Hence $\operatorname{coeff}_x(r_x\gamma) < c_m$, and thus $\operatorname{coeff}_x(r_x\gamma) = 1$ by Proposition (2.2)(ii). If x and y_1 are adjoined by a non-simple bond, then $\langle x, y \rangle = -\cos(\pi/m)$, and Corollary (4.5) yields that $\operatorname{coeff}_{y_1}(r_x\gamma) \geq c_m$. If x and y_1 are adjoined by a simple bond, then $\operatorname{coeff}_{y_1}(\gamma) \neq 1$ by Corollary (4.5), and therefore $\operatorname{coeff}_{y_1}(r_x\gamma) \geq 2$ by Corollary (4.5). Hence in any case,

$$\operatorname{coeff}_{y_1}(\gamma)\langle x, y_1 \rangle = \operatorname{coeff}_{y_1}(r_x\gamma)\langle x, y_1 \rangle \leq -1.$$

Similarly, $\operatorname{coeff}_{y_2}(\gamma)\langle x, y_2 \rangle \leq -1$. Since $\operatorname{coeff}_z(\gamma)\langle x, z \rangle \leq 0$ for $z \neq x, y_1, y_2$, this forces

$$\langle x, \gamma \rangle = \sum_{z \in \operatorname{supp}(\gamma)} \operatorname{coeff}_z(\gamma)\langle x, z \rangle \leq c_m + (-1) + (-1) < 0,$$

contradicting our earlier conclusion that $\langle x, \gamma \rangle > 0$. Therefore x is adjoined to at most one vertex in $\operatorname{supp}(\gamma)$. Connectedness of $\Gamma(\gamma)$ now yields that $\operatorname{supp}(\gamma) \subseteq S_0$ or $\operatorname{supp}(\gamma) \subseteq S_1$. The latter is impossible, as the coefficients of roots in Φ_{S_1} are integers, and $\operatorname{coeff}_x(\gamma)$ is not; hence $\operatorname{supp}(\gamma) \subseteq S_0$.

Since $\gamma \preceq \alpha$, and the coefficients of x in γ and α coincide, there exists a $w \in W_{S \setminus \{x\}}$ with $\alpha = w\gamma$ and $\operatorname{dp}(\alpha) = l(w) + \operatorname{dp}(\gamma)$. Now $W_{S \setminus \{x\}}$ is the direct product of $W_{S_0 \setminus \{x\}}$ and $W_{S_1'}$; thus $w = w_0 w_1$ for some $w_0 \in W_{S_0 \setminus \{x\}}$ and $w_1 \in W_{S_1'}$ with $l(w) = l(w_0) + l(w_1)$. Define $\beta = w_0\gamma$; then $\beta \in \Phi_{S_0}^+$ with $\operatorname{coeff}_x(\beta) = \operatorname{coeff}_x(\gamma) = c_m$. Further, $\alpha = w_1\beta = \phi(\beta, w_1 x)$ by $(*)$; hence ϕ is onto.

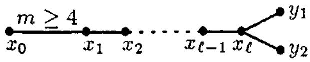
It remains to show that ϕ restricts to a one-one correspondence between the given sets. Let w_1 be in $W_{S_1'}$ and β be in Φ_{S_0} with $\operatorname{coeff}_x(\beta) = c_m$; it suffices to show that $w_1\beta \notin \mathscr{E}$ if and only if $\beta \notin \mathscr{E}$ or $w_1 x \notin \mathscr{E}$. Observe that $(*)$ yields for $\delta \in \Phi_{S_1'}^+$ that

$$\begin{aligned}\langle w_1\beta, \delta \rangle &= \langle \beta + c_m(w_1x - x), \delta \rangle \\ &= \langle \beta - c_m x, \delta \rangle + c_m\langle w_1 x, \delta \rangle = c_m\langle w_1 x, \delta \rangle. \qquad (**)\end{aligned}$$

It can be easily seen that $w_1\beta$ is preceded by β; thus $w_1\beta \notin \mathscr{E}$ by Proposition (2.2)(iii) if $\beta \notin \mathscr{E}$. Next, suppose that $w_1 x$ dominates $\delta \in \Phi^+ \setminus \{w_1 x\}$; then $\delta \in \Phi_{S_1'}$ as $x \in \mathscr{E}$. Since $\langle w_1 x, \delta \rangle \geq 1$ by Proposition (2.2)(i), Eq. $(**)$ yields that $\langle w_1\beta, \delta \rangle \geq 1$. Therefore $w_1\beta \operatorname{dom} \delta$ or $\delta \operatorname{dom} w_1\beta$

by Proposition (2.2)(i). But δ cannot dominate $w_1\beta$, as $\operatorname{supp}(\delta) \subset \operatorname{supp}(w_1\beta)$; hence $w_1\beta \notin \mathscr{E}$, as required.

Conversely, if $w_1\beta$ dominates some $\delta \in \Phi^+ \setminus \{w_1\beta\}$, Proposition (2.2)(ii) implies that either $\beta \operatorname{dom} w_1^{-1}\delta$, or $w_1^{-1}\delta$ is negative. So $\beta \notin \mathscr{E}$ (as required) or $\delta \in \Phi^+_{S_1'}$. If $\delta \in \Phi^+_{S_1'}$, Proposition (2.2)(i) and $(**)$ imply that $\langle w_1 x, \delta \rangle \geq \frac{1}{c_m} > \frac{1}{2}$. As $\Gamma(S_1)$ contains only simple bonds, this forces $\langle w_1 x, \delta \rangle \geq 1$; hence $\delta \operatorname{dom} w_1 x$ or $w_1 x \operatorname{dom} \delta$. Since $x \notin \operatorname{supp}(\delta)$ while $x \in \operatorname{supp}(w_1 x)$, the former is impossible, and thus $w_1 x \notin \mathscr{E}$, as required. ∎



(6.2) LEMMA. *Let α be a root such that $\Gamma(\alpha)$ contains the subgraph with* $\operatorname{coeff}_{x_0}(\alpha) > 1$ *and* $\operatorname{coeff}_{x_1}(\alpha), \dots, \operatorname{coeff}_{x_l}(\alpha) \geq 2$. *Then* $\alpha \notin \mathscr{E}$.

Proof. If $\Gamma(\alpha)$ contains a circuit or an infinite bond, the assertion is certainly true by Lemma (4.1). Suppose next that $\Gamma(\alpha)$ is a tree which contains only finite weights. By Proposition (4.2) and Lemma (4.6), we may assume without loss of generality that $\Gamma(\alpha)$ contains exactly one non-simple bond (namely, the one of weight m with vertices x_0 and x_1).

As in the proof of Lemma (4.1), let $\beta \preceq \alpha$ be minimal subject to the following conditions: $y_1, y_2 \in \operatorname{supp}(\beta)$, the coefficient λ_0 of x_0 in β is greater than 1, and the coefficient λ_i of x_i in β is greater than or equal to 2 for $i = 1, \dots, l$.

Next, let $z \in \Pi$ with $r_z\beta \prec \beta$. Once again, it suffices to show that $\langle r_z\beta, z \rangle \leq -1$. Minimality of β forces $z = y_1, y_2$, or $z = x_i$ for some $i \in \{0, \dots, l\}$.

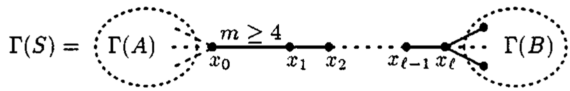
If $z = y_j$, minimality of β yields that $y_j \notin \operatorname{supp}(r_{y_j}\beta)$; since $\lambda_l \geq 2$, we deduce that $\langle r_{y_j}\beta, y_j \rangle \leq -1$.

Suppose next that $z = x_0$; then minimality of β together with Proposition (2.1)(i) yields that $\operatorname{coeff}_{x_0}(r_{x_0}\beta) = 0$ or 1. In the former case, it follows easily that $\langle r_{x_0}\beta, x_0 \rangle \leq -1$. If $\operatorname{coeff}_{x_0}(r_{x_0}\beta) = 1$, Lemma (4.4) yields that $r_{x_0}\beta = wx_0$ for some $w \in W_{S'}$, where $S' := \operatorname{supp}(r_{x_0}\beta) \setminus \{x_0\}$. Let $\Gamma(S_1')$ be the connected component of $\Gamma(S')$ which contains x_1; then $w = w_1 w_2$ with $w_1 \in W_{S_1'}$ and $w_2 \in W_{S' \setminus S_1'}$, and an easy induction on $l(w_1)$ yields that $\operatorname{coeff}_x(w_1 x_0)$ is an integer multiple of c_m for all $x \in S_1$. In particular, $\lambda_1 = \operatorname{coeff}_{x_1}(r_{x_0}\beta) = \operatorname{coeff}_{x_1}(w_1 x_0) \in c_m\mathbb{N}$. Since $\lambda_1 \geq 2$, we deduce that $\lambda_1 \geq 2c_m$; whence again, $\langle r_{x_0}\beta, x_0 \rangle \leq -1$.

Finally, if $z = x_i$ for some $i \in \{1, \ldots, l\}$, minimality of β together with Proposition (2.1)(ii) yields that the coefficient λ_i' of x_i in $r_{x_i}\beta$ equals 0, 1, or c_m. If $\lambda_i' \leq 1$, it can be easily seen that $\langle r_{x_i}\beta, x_i\rangle \leq -1$; so suppose that $\lambda_i' = c_m$. If $i < l$, the coefficient of x_{i+1} in $r_{x_i}\beta$ is an integer multiple of c_m by Proposition (6.1), and thus greater than or equal to $2c_m$; if $i = l$, the coefficients of y_1 and y_2 in $r_{x_i}\beta$ are integer multiples of c_m by Proposition (6.1). Thus in any case $\langle r_{x_i}\beta, x_i\rangle \leq -1$, as required. ∎

The next assertion follows immediately from Proposition (6.1) together with the previous lemma.

(6.3) PROPOSITION. *Suppose that $\Gamma(S)$ is a finite tree and contains exactly one non-simple bond, which has finite weight m. Suppose further that*



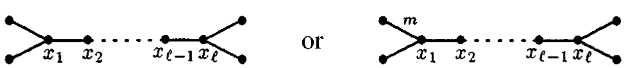
with disjoint sets $A, B \subseteq S$ such that $\Gamma(A \cup \{x_0\})$ and $\Gamma(B \cup \{x_l\})$ are connected, and x_l is adjoined to two or more vertices of $\Gamma(B)$. Then the elements of $\mathscr{E}_S$ are as follows (some may be listed more than once):

(a) $\alpha_i + \beta_i - x_i$, *where* $i \in \{0, \ldots, l\}$ *and* $\alpha_i \in \mathscr{E}_{A \cup \{x_0, \ldots, x_i\}}$, $\beta_i \in \mathscr{E}_{\{x_i, \ldots, x_l\} \cup B}$ *have coefficient* 1 *for* x_i,

(b) $\alpha_i + c_m\beta_i - c_m x_i$, *where* $i \in \{1, \ldots, l\}$, $\alpha_i \in \mathscr{E}_{A \cup \{x_0, \ldots, x_i\}}$ *has coefficient* c_m *for* x_i, *and* $\beta_i \in \mathscr{E}_{\{x_i, \ldots, x_l\} \cup B}$ *has coefficient* 1 *for* x_i.

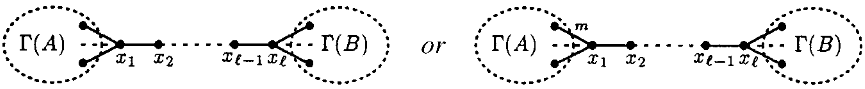
Note that $\{x_i, \ldots, x_l\} \cup B$ defined above is a proper subset of S unless $A = \emptyset$ and $i = 0$, and $A \cup \{x_0, \ldots, x_i\}$ is a proper subset of S for all i. Therefore Proposition (6.3) gives an inductive description of $\mathscr{E}_S$ for those $S \subseteq \Pi$ with $\Gamma(S)$ of the shape described above with $A \neq \emptyset$.

Arguments similar to the ones applied in the proof of Lemma (6.2) yield that a root α is not elementary if its support contains one of the subgraphs



(with $l \geq 1$), and $\operatorname{coeff}_{x_i}(\alpha) \geq 2$ for $i = 1, \ldots, l$ (compare Lemma (5.1)). Together with Proposition (6.1), this yields the next assertion, which is a "variation" of Proposition (5.2).

(6.4) PROPOSITION. *Suppose that $\Gamma(S)$ is a finite tree and contains exactly one non-simple bond, which has finite weight m. Suppose further that $\Gamma(S)$ equals*



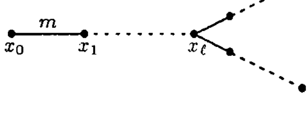
with $l \geq 1$ and disjoint sets $A, B \subseteq S$ such that $\Gamma(A \cup \{x_1\})$ and $\Gamma(\{x_l\} \cup B)$ are connected, x_1 is adjoined to at least two vertices of $\Gamma(A)$, and x_l is adjoined to at least two vertices of $\Gamma(B)$. Finally, suppose that $\Gamma(\{x_l\} \cup B)$ contains only simple bonds. Then the elements of $\mathscr{E}_S$ are as follows (some may be listed more than once):

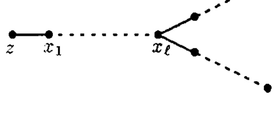
(a) $\alpha_i + \beta_i - x_i$, *where* $i \in \{1, \ldots, l\}$ *and* $\alpha_i \in \mathscr{E}_{A \cup \{x_1, \ldots, x_i\}}$, $\beta_i \in \mathscr{E}_{\{x_i, \ldots, x_l\} \cup B}$ *both have coefficient* 1 *for* x_i, *and*

(b) $\alpha_i + c_m \beta_i - c_m x_i$, *where* $i \in \{1, \ldots, l\}$, $\beta_i \in \mathscr{E}_{\{x_i, \ldots, x_l\} \cup B}$ *has coefficient* 1 *for* x_i, *and* $\alpha_i \in \mathscr{E}_{A \cup \{x_1, \ldots, x_i\}}$ *has coefficient* c_m *for* x_i.

Since $\mathscr{E}_{S'}$ for $S' \subset S$ is already known by the inductive hypothesis, Propositions (6.3) and (6.4) leave us to determine $\mathscr{E}_S$ for $S \subseteq \Pi$ with $\Gamma(S)$ of shapes (1) and (2), as defined in Section 4. The next proposition together with Lemma (6.2) will enable us to determine $\mathscr{E}_S$ for $S \subseteq \Pi$ with $\Gamma(S)$ of shape (2) from $\mathscr{E}_T$ for T with $\Gamma(T)$ of shape (0) or (1).

(6.5) PROPOSITION. *Let $S \subseteq \Pi$ be such that $\Gamma(S)$ is of shape (2). Since $\mathscr{E}_S$ does not depend on $\Pi \setminus S$, we may assume without loss of generality that there exists a $z \in \Pi \setminus S$ such that $m_{r_z r_y} = 2$ for all $y \in S \setminus \{x_1\}$ and $m_{r_z r_{x_1}} = 3$. Define $\tilde{S} = (S \setminus \{x_0\}) \cup \{z\}$. That is,*

$\Gamma(S) =$ 

and $\Gamma(\tilde{S}) =$ 

Then ϕ: $\{\beta \in \Phi_{\tilde{S}}^{+} \mid \operatorname{coeff}_z(\beta) = 1\} \to \{\alpha \in \Phi_S^+ \mid \operatorname{coeff}_{x_0}(\alpha) = 1\}$, $\beta \mapsto x_0 + c_m(\beta - z)$ *defines a one-one correspondence. Moreover,* ϕ *restricts to a one-one correspondence*

$$\{\beta \in \mathscr{E}_{\tilde{S}} \mid \operatorname{coeff}_z(\beta) = 1\} \leftrightarrow \{\alpha \in \mathscr{E}_S \mid \operatorname{coeff}_{x_0}(\alpha) = 1\}.$$

Furthermore, $\operatorname{dp}(\phi(\beta)) = \operatorname{dp}(\beta)$ *for all* $\beta \in \Phi_{\tilde{S}}^{+}$ *with* $\operatorname{coeff}_z(\beta) = 1$.

Note that the correspondences given above hold whenever $\Gamma(S)$ is a tree such that $\Gamma(S \setminus \{x_0\})$ contains only simple bonds and $\Gamma(\tilde{S})$ is the graph obtained from $\Gamma(S)$ by replacing all non-simple bonds by simple ones.

Proof of (6.5). Since $\langle x_0, x_1\rangle = c_m\langle z, x_1\rangle$ and $\langle x_0, y\rangle = 0 = \langle z, y\rangle$ for $y \in S \setminus \{x_0, x_1\}$, clearly $\langle x_0, y\rangle = c_m\langle z, y\rangle$ for $y \in S \setminus \{x_0\}$. This yields for an arbitrary root β and y in $S \setminus \{x_0\}$ that

$$r_y(x_0 + c_m(\beta - z)) = x_0 + c_m(r_y\beta - z). \qquad (*)$$

To show that ϕ is well defined, let β be in $\Phi_{\tilde{S}}^{+}$ with coefficient 1 for z. In accordance with Lemma (4.4), let $w \in W_{S\setminus\{x_0\}}$ be of length $\operatorname{dp}(\beta) - 1$ such that $\beta = wz$. Then $\operatorname{coeff}_{x_0}(wx_0) = 1$. Since $x_0 = x_0 + c_m(z - z)$, a straightforward induction on $l(w)$ using $(*)$ yields that $wx_0 = x_0 + c_m(wz - z)$ and $\operatorname{dp}(wx_0) = \operatorname{dp}(wz)$; hence ϕ is well defined and preserves depth. By Lemma (4.4), every element of Φ_S^+ with coefficient 1 for x_0 can be written as wx_0 for some $w \in W_{S\setminus\{x_0\}}$; so the above also shows that ϕ is onto. Since ϕ is certainly one-one, ϕ is a one-one correspondence. As in the proof of Proposition (6.1), it follows readily that ϕ restricts to a one-one correspondence between the set of roots in $\mathscr{E}_{\tilde{S}}$ with coefficient 1 for z, and the set of roots in $\mathscr{E}_S$ with coefficient 1 for x_0. ∎

(6.6) PROPOSITION. *Let* $S \subseteq \Pi$ *be such that* $\Gamma(S)$ *is of shape* (2). *Then the elements of* $\mathscr{E}_S$ *are as follows* (*some may be listed more than once*):

(a) $\alpha_i + \beta_i - x_i$, *where* $i \in \{1, \dots, l\}$ *and* $\alpha_i \in \mathscr{E}_{\{x_0, \dots, x_i\}}$, $\beta_i \in \mathscr{E}_{S\setminus\{x_0, \dots, x_{i-1}\}}$ *have coefficient* 1 *for* x_i,

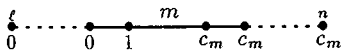
(b) $x_0 + c_m(\alpha_0 - z)$, *where* $\alpha_0 \in \mathscr{E}_{\tilde{S}}$ *has coefficient* 1 *for* z (*and* z, $\tilde{S}$ *are as in Proposition* (6.5)), *and*

(c) $\alpha_i + c_m\beta_i - c_m x_i$, *where* $i \in \{1, \dots, l\}$, $\alpha_i \in \mathscr{E}_{\{x_0, \dots, x_i\}}$ *has coefficient* c_m *for* x_i, *and* $\beta_i \in \mathscr{E}_{S\setminus\{x_0, \dots, x_{i-1}\}}$ *has coefficient* 1 *for* x_i.

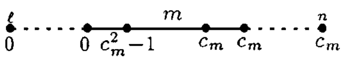
Observe that $\Gamma(\tilde{S})$ and $\Gamma(S \setminus \{x_0, \dots, x_{i-1}\})$ defined above are of shape (0), and thus $\mathscr{E}_{\tilde{S}}$ and $\mathscr{E}_{S\setminus\{x_0, \dots, x_{i-1}\}}$ have already been determined in Section 5. Since $\Gamma(\{x_0, \dots, x_i\})$ is of shape (1), this leaves us now to determine $\mathscr{E}_S$ for $S \subseteq \Pi$ with $\Gamma(S)$ of shape (1).

From now on, suppose that $\Gamma(S)$ is of shape (1). As in Section 5 for $S \subseteq \Pi$ with $\Gamma(S)$ of shape (0), we shall give roots which precede the elements of $\mathscr{E}_S$; this will enable us to compile a complete list of elements of $\mathscr{E}_S$.

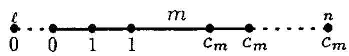
Since $\Gamma(S)$ is of shape (1), every root in Φ_S is preceded by some x_i or y_j. It follows readily by Proposition (2.2)(v) together with Lemma (4.3) that each element of $\mathscr{E}_S$ must be preceded by $x_1 + c_m y_1$ or $c_m x_1 + y_1$. If $\alpha \in \mathscr{E}_S$ is preceded by $x_1 + c_m y_1$, a straightforward induction on n using Proposition (2.2)(v) and Lemma (4.3) yields that α is also preceded by $\alpha_{1,n} := (r_{y_n} \cdots r_{y_2} r_{y_1}) x_1 = x_1 + c_m(y_1 + \cdots + y_n)$, and that this is an elementary root. We denote $\alpha_{1,n}$ by the diagram



Since $r_{x_i} \alpha_{1,n} = \alpha_{1,n}$ for $i \geq 3$ and $r_{y_j} \alpha_{1,n} \preceq \alpha_{1,n}$ for all j, it follows that each root in $\mathscr{E}_S$ which is preceded by, but not equal to, $\alpha_{1,n}$, is preceded by



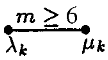
or



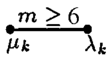
(with $l \geq 2$ in the latter case), which we denote by $\delta_{1,n}$ and $\alpha_{2,n}$, respectively.

Suppose now that $m \geq 6$. Then $\langle \delta_{1,n}, x_2 \rangle \leq -1$, and thus $\operatorname{coeff}_{x_2}(\beta) = 0$ for each elementary root β preceded by $\delta_{1,n}$ by Proposition (2.2)(vi). In particular, no root in $\mathscr{E}_S$ can be preceded by $\delta_{1,n}$ if $l \geq 2$; therefore, each root in $\mathscr{E}_S$ preceded by $\alpha_{1,n}$ must also be preceded by $\alpha_{2,n}$ if $l \geq 2$. Since $\langle \alpha_{2,n}, x_1 \rangle \leq -1$, Proposition (2.2)(vi) further yields that $\operatorname{coeff}_{x_1}(\alpha) = 1$ for each $\alpha \in \mathscr{E}_S$ preceded by $\alpha_{2,n}$. If $l = 1$, then $\alpha_{1,n} \in \mathscr{E}_S$, and a straightforward induction on depth yields a complete list of the elements of $\mathscr{E}_S$ preceded by $\alpha_{1,n}$. We can thus deduce the following assertion.

(6.7) PROPOSITION. *Let $S \subseteq \Pi$ be such that $\Gamma(S)$ is of shape* (1). *If $m \geq 6$ is even and $l = n = 1$, the elements of $\mathscr{E}_S$ are*



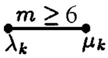
$(k = 1, \ldots, \frac{m-2}{2})$ *and*



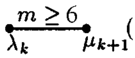
$(k = 1, \ldots, \frac{m-2}{2})$,

where $\lambda_k = \sin((2k+1)\pi/m)/\sin(\pi/m)$ *and* $\mu_k = \sin(2k\pi/m)/\sin(\pi/m)$ *for* $k \in \mathbb{N}_0$.

If $m \geq 6$ *is odd and* $l = n = 1$, *the elements of* $\mathscr{E}_S$ *are*



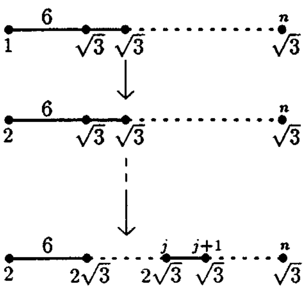
$(k = 1, \ldots, \frac{m-3}{2})$ *and*



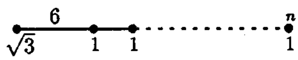
$(k = 0, \ldots, \frac{m-3}{2})$

with λ_k, μ_k *as above.*

If $m = 6$, $l = 1$, *and* $n \geq 2$, *the elements of* $\mathscr{E}_S$ *are*

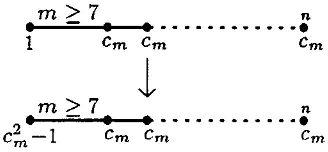


and

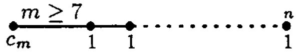


$(1 \leq j \leq n-1)$.

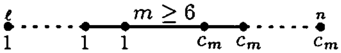
If $m \geq 7$, $l = 1$, *and* $n \geq 2$, *the elements of* $\mathscr{E}_S$ *are*



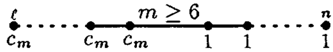
and



If $m \geq 6$ *and* $l, n \geq 2$, *the elements of* $\mathscr{E}_S$ *are*

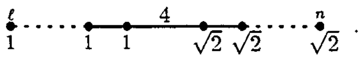


and

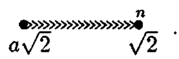


.

This leaves us to determine $\mathscr{E}_S$ for $\Gamma(S)$ of shape (1) as defined in Section 3 with $m \in \{4, 5\}$. Suppose first that $m = 4$; then $c_m = \sqrt{2}$. If $\alpha \in \Phi_S$ is preceded by $\alpha_{1,n}$, a straightforward induction on depth yields that the coefficients of the x_i in α are integers, and the coefficients of the y_j in α are integer multiples of $\sqrt{2}$. A straightforward induction on l (using Proposition (2.2)(v) and Lemma (4.3)) yields that each $\alpha \in \mathscr{E}_S$ which is preceded by $\alpha_{1,n}$ is also preceded by

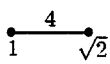
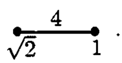


In accordance with the notation introduced in Section 5, we denote for $a \in \mathbb{N}$ sums $\sum_{j=1}^{n}(\mu_j\sqrt{2})y_j$ with $\mu_1 = a$, $\mu_n = 1$, and $\mu_{j+1} \in \{\mu_j, \mu_j - 1\}$ for all $j \leq n - 1$ by the diagram

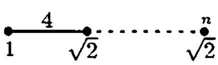
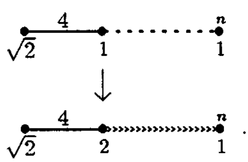


Since predecessors of elementary roots are elementary by Proposition (2.2)(iii), straightforward inductions on depth (using Proposition (2.2)(v)) now yield the following result.

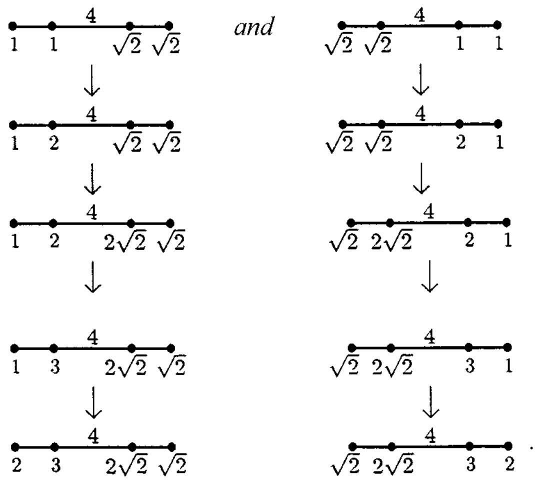
(6.8) PROPOSITION. *Suppose that S is of shape* (1) *with* $m = 4$.
If $l = n = 1$, *the elements of* $\mathscr{E}_S$ *are*

 *and* 

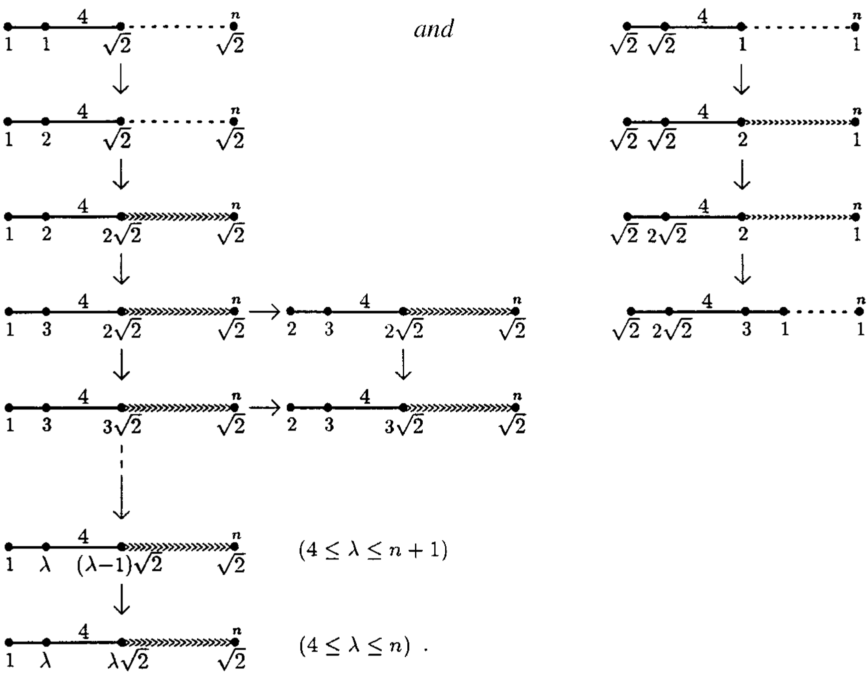
If $l = 1$ *and* $n \geq 2$, *the elements of* $\mathscr{E}_S$ *are*

 *and* 

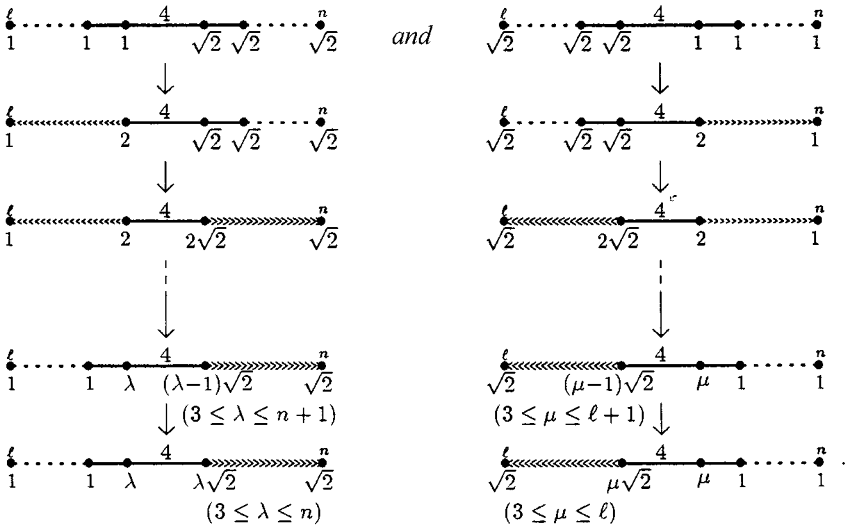
If $l = n = 2$, the elements of $\mathscr{E}_S$ are



If $l = 2$ and $n \geq 3$, the elements of $\mathscr{E}_S$ are



If l, $n \geq 3$, the roots in $\mathscr{E}_S$ are

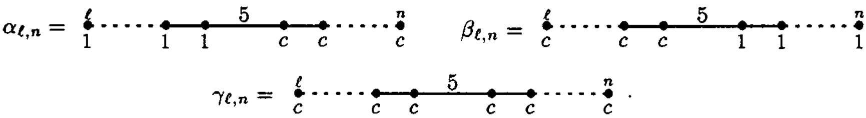


It remains to determine $\mathscr{E}_S$ for $S \subseteq \Pi$ with $\Gamma(S)$ of shape (1) and $m = 5$. We define $c := c_5 = 2\cos(\frac{\pi}{5}) = \frac{1+\sqrt{5}}{2}$. Since $c^2 = c + 1$, Proposition (2.1)(ii) yields that the coefficients of simple roots in any element of Φ_S^+ are of the form $a + bc$ with $a, b \in \mathbb{N}_0$; in particular, the smallest possible positive coefficients are $1 < c < 2 < c + 1 < 3$. Observe also that $\Phi^+_{\{x_l,\dots,x_1\}} = \{x_i + \cdots + x_j \mid l \geq i \geq j \geq 1\}$. So if $\mathrm{coeff}_{x_i}(\alpha) = 1$ for some root $\alpha \in \Phi_S$, then $\mathrm{coeff}_{x_{i+1}}(\alpha) \in \{0, 1\}$ by Proposition (4.2); similarly, if $\mathrm{coeff}_{x_i}(\alpha) = c$ for some root $\alpha \in \Phi_S$, then $\mathrm{coeff}_{x_{i+1}}(\alpha) \in \{0, c\}$ by Proposition (6.1). Therefore:

$$\text{If } \mathrm{coeff}_{x_{i+1}}(\alpha) > 0 \text{ and } \mathrm{coeff}_{x_i}(\alpha) \in \{1, c\},$$

$$\text{then } \mathrm{coeff}_{x_{i+1}}(\alpha) = \mathrm{coeff}_{x_i}(\alpha). \quad (*)$$

(6.9) LEMMA. *Let $S \subseteq \Pi$ be such that $\Gamma(S)$ is of shape* (1) *with $m = 5$. Then each root in $\mathscr{E}_S$ is preceded by one of the roots*



In order to prove Lemma (6.9), we establish the following two results.

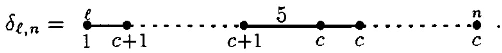
(6.10) LEMMA. *Let $S \subseteq \Pi$ be such that $\Gamma(S)$ is of shape* (1) *with $m = 5$. Further, let $\alpha \in \Phi_S$ and $i \in \{1, \dots, l-1\}$ be such that* $\operatorname{coeff}_{x_i}(\alpha) = 2$ *and* $\operatorname{coeff}_{x_{i+1}}(\alpha) > 0$. *Then $\alpha \notin \mathscr{E}$.*

Proof. Let $\beta \preceq \alpha$ be minimal such that there exists some $k \in \{1, \dots, l-1\}$ with $x_{k+1} \in \operatorname{supp}(\beta)$ and $\operatorname{coeff}_{x_k}(\beta) = 2$. Once more, it suffices to show that $\langle r_z \beta, z \rangle \leq -1$, where $z \in S$ with $r_z \beta \prec \beta$. By minimality of β, we know that $z \in \{x_{k+1}, x_k\}$.

If $z = x_{k+1}$, minimality of β forces $x_{k+1} \notin \operatorname{supp}(r_{x_{k+1}} \beta)$, and it follows easily that $\langle r_{x_{k+1}} \beta, x_{k+1} \rangle \leq -1$.

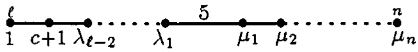
Assume for a contradiction that $z = x_k$; then the coefficient λ'_k of x_k in $r_{x_k} \beta$ equals $0, 1$, or c. Since $\operatorname{coeff}_{x_{k+1}}(\beta) \neq 0$, and clearly $\operatorname{supp}(\beta) \not\subseteq \{x_l, \dots, x_{k+1}, x_k\}$, connectedness of $\Gamma(r_{x_k} \beta)$ yields that $\lambda'_k \in \{1, c\}$; hence $\operatorname{coeff}_{x_{k+1}}(\beta) = \lambda'_k$ by $(*)$. If $k > 1$, we deduce that $\operatorname{coeff}_{x_{k-1}}(\beta) = 2$; but as $x_k \in \operatorname{supp}(r_{x_k} \beta)$, this contradicts the minimality of β. If $k = 1$, it follows that $c \cdot \operatorname{coeff}_{y_1}(\beta) = 2$, contradicting the fact that $\operatorname{coeff}_{y_1}(\beta) = a + bc$ with $a, b \in \mathbb{N}_0$, and $c(a + bc) = b + (a + b)c \neq 2$. ∎

(6.11) LEMMA. *Let $S \subseteq \Pi$ be such that $\Gamma(S)$ is of shape* (1) *with $m = 5$ and $l \geq 2$. Further, let $\alpha \in \mathscr{E}_S$ with* $\operatorname{coeff}_{x_{l-1}}(\alpha) = c + 1$. *Then α is preceded by*



Proof. If $l = 2$, $n = 1$ the assertion can be easily verified; so suppose that $l \geq 2$, $n \geq 1$, and $l + n > 3$, and proceed by induction on $l + n$. Let $\beta \preceq \alpha$ be minimal such that $\beta \in \mathscr{E}_S$ and $\operatorname{coeff}_{x_{l-1}}(\beta) = c + 1$. By transitivity of $\preceq$, it suffices to show that β is preceded by $\delta_{l,n}$.

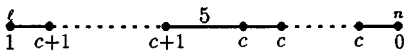
Denote the coefficient of x_l in β by λ_l. Then $\lambda_l < c + 1$, as $\langle \beta, x_l \rangle < 1$; further, $\lambda_l > 0$ by construction. Since $\operatorname{coeff}_{x_l}(r_{x_l}\beta) = -\lambda_l + c + 1$ has to be in $\mathbb{N}_0 + c\mathbb{N}_0$, we find that $\lambda_l \in \{1, c\}$. If $\lambda_l = c$, then $r_{x_l}\beta \prec \beta$ and $\operatorname{supp}(r_{x_l}\beta) = S$, contradicting the minimality of β. Therefore $\lambda_l = 1$; that is,

$\beta =$ 

.

Now let $z \in S$ with $r_z\beta \prec \beta$; clearly $z \neq x_l$, and thus $z \in \{x_{l-1}, y_n\}$ by minimality of β.

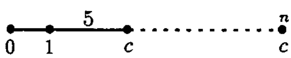
If $z = y_n$, minimality of β yields that $y_n \notin \operatorname{supp}(r_{y_n}\beta)$; hence $r_{y_n}\beta \in \mathscr{E}_{S\setminus\{y_n\}}$. Since $\operatorname{coeff}_{x_{l-1}}(r_{y_n}\beta) = c + 1$, clearly $\operatorname{supp}(r_{y_n}\beta) \not\subseteq \{x_l, \ldots, x_1\}$; therefore $n \geq 2$. By the inductive hypothesis, $r_{y_n}\beta$ is preceded by

$\delta_{\ell,n-1} =$ 

.

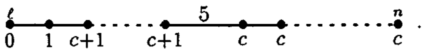
This yields that $\mu_{n-1} \geq c$. If $\mu_{n-1} \geq 2$, then $y_n \notin \operatorname{supp}(r_{y_n}\beta)$ forces $\langle r_{y_n}\beta, y_n \rangle \leq -1$, contradicting $\beta \in \mathscr{E}$. Therefore $\mu_{n-1} = c$, and thus $\mu_n = c$ by $(*)$. By Lemma (4.3), we find that β is preceded by $r_{y_n}(\delta_{l,n-1}) = \delta_{l,n}$, as required.

Assume next that $z = x_{l-1}$. Then the coefficient λ'_{l-1} of x_{l-1} in $r_{x_{l-1}}\beta$ is less than $c + 1$, and thus $\lambda'_{l-1} \in \{0, 1, c, 2\}$. But $\lambda'_{l-1} \neq 2$ by the previous lemma, $\lambda'_{l-1} \neq c$ by $(*)$, and $\lambda'_{l-1} \neq 0$ by connectedness of $\Gamma(r_{x_{l-1}}\beta)$. Therefore $\lambda'_{l-1} = 1$. This forces $\lambda_{l-2} = c + 1$ if $l > 2$, and $\mu_1 = c$ if $l = 2$; furthermore, $r_{x_l}r_{x_{l-1}}\beta \in \mathscr{E}_{S\setminus\{x_l\}}$. If $l = 2$, then $(*)$ (applied to $y_1, \ldots, y_n$) yields that

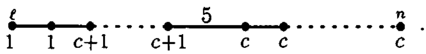
$r_{x_\ell}r_{x_{\ell-1}}\beta =$ 

.

Hence $\beta = \delta_{2,n}$, which is certainly preceded by $\delta_{2,n}$. If $l > 2$, the inductive hypothesis yields that $r_{x_l} r_{x_{l-1}} \beta$ is preceded by

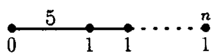
$\delta_{\ell-1,n} =$ 

Since $\lambda'_{l-1} = 1$, Lemma (4.3) gives that $r_{x_{l-1}} \beta$ is preceded by

$r_{x_\ell}(\delta_{\ell-1,n}) =$ 

As $\lambda_{l-2} = c + 1$, Lemma (4.3) further yields that β is preceded by $r_{x_{l-1}} r_{x_l}(\delta_{l-1,n}) = \delta_{l,n}$, as required. ∎

Proof of (6.9). If $l = n = 1$, the assertion can be easily verified; so assume that $l, n \geq 1$ with $l + n > 2$, and proceed by induction on $l + n$. Suppose that $\alpha \in \mathscr{E}_S$, and let $\beta \preceq \alpha$ be minimal with $\operatorname{supp}(\beta) = S$. It suffices to show that β is preceded by one of $\alpha_{l,n}$, $\beta_{l,n}$, or $\gamma_{l,n}$. Let $x \in S$ with $r_x \beta \prec \beta$. Then $x \in \{x_l, y_n\}$ by minimality of β. By symmetry, we may assume without loss of generality that $x = x_l$. Then $r_{x_l} \beta \in \mathscr{E}_{S \setminus \{x_l\}}$ by minimality of β.

$r_{x_\ell}\beta =$ 

If $l = 1$, then
hence $\beta = \beta_{1,n}$, as required. Suppose next that $l > 1$; then the inductive hypothesis applies to $r_{x_l} \beta$, and this is preceded by $\alpha_{l-1,n}$, $\beta_{l-1,n}$ or $\gamma_{l-1,n}$. Since the coefficient of x_l in $r_{x_l} \beta$ is 0, the coefficients of x_l and x_{l-1} in β coincide; denote this coefficient by λ_{l-1}. As $\langle \beta, x_l \rangle < 1$, we know that $\lambda_{l-1} \in \{1, c\}$. If $r_{x_l} \beta$ is preceded by $\beta_{l-1,n}$ or $\gamma_{l-1,n}$, it follows that $\lambda_{l-1} \geq c$; that is, $\lambda_{l-1} = c$. Now Lemma (4.3) yields that β is preceded by $r_{x_l} \beta_{l-1,n} = \beta_{l,n}$ or $r_{x_l} \gamma_{l-1,n} = \gamma_{l,n}$. If $r_{x_l} \beta$ is preceded by $\alpha_{l-1,n}$ and

$\lambda_{l-1} = 1$, we deduce from Lemma (4.3) that β is preceded by $r_{x_l}\alpha_{l-1,n} = \alpha_{l,n}$, as required. Hence it remains to consider the case that $\alpha_{l-1,n} \preceq r_{x_l}\beta$ and $\lambda_{l-1} = c$.

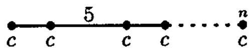
Let γ be maximal such that $\alpha_{l-1,n} \preceq \gamma \preceq r_{x_l}\beta$ and $\operatorname{coeff}_{x_{l-1}}(\gamma) = 1$. Maximality of γ yields that $\gamma \prec r_{x_{l-1}}\gamma \preceq r_{x_l}\beta$. Hence

$$1 < \operatorname{coeff}_{x_{l-1}}(r_{x_{l-1}}\gamma) \leq \operatorname{coeff}_{x_{l-1}}\big(r_{x_l}\beta\big) = c,$$

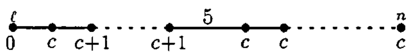
and therefore $\operatorname{coeff}_{x_{l-1}}(r_{x_{l-1}}\gamma) = c$. Whence either $l = 2$ and $\operatorname{coeff}_{y_1}(\gamma) = c$, or $l > 2$ and $\operatorname{coeff}_{x_{l-2}}(\gamma) = c + 1$. In the former case, $(*)$ gives that

$r_{x_1}\gamma =$ 

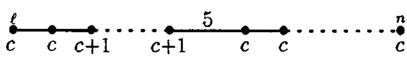
and it follows by Lemma (4.3) that β is preceded by

$\gamma_{2,n} = r_{x_2}(r_{x_1}\gamma) =$ 

as required. If $l > 2$, Lemma (6.11) implies that γ is preceded by $\delta_{l-1,n}$ (as defined in Lemma (6.11)). Since $\operatorname{coeff}_{x_{l-2}}(\gamma) = c + 1$, Lemma (4.3) yields that $r_{x_{l-1}}\gamma$ (and thus $r_{x_l}\beta$) is preceded by

$r_{x_{\ell-1}}(\delta_{\ell-1,n}) =$  .

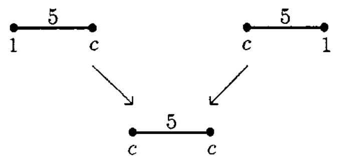
As $\lambda_{l-1} = c$, it follows—again—by Lemma (4.3) that β is preceded by

$r_{x_\ell}(r_{x_{\ell-1}}\delta_{\ell-1,n}) =$ 

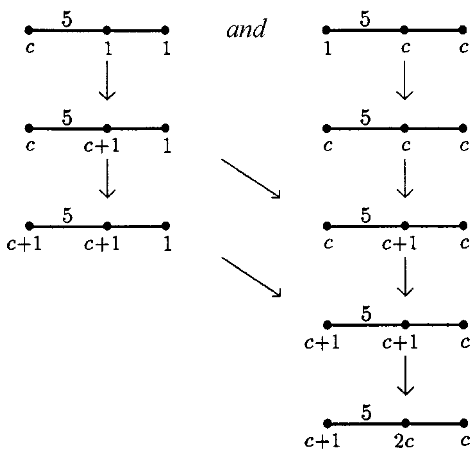
which is clearly preceded by $\gamma_{l,n}$, and this completes the proof. ∎

By Lemma (6.9), all roots in $\mathscr{E}_S$ are preceded by at least one of $\alpha_{l,n}$, $\beta_{l,n}$, and $\delta_{l,n}$. Since all predecessors of elementary roots are elementary by Proposition (2.2)(iii), straightforward inductions on depth now yield the following assertion.

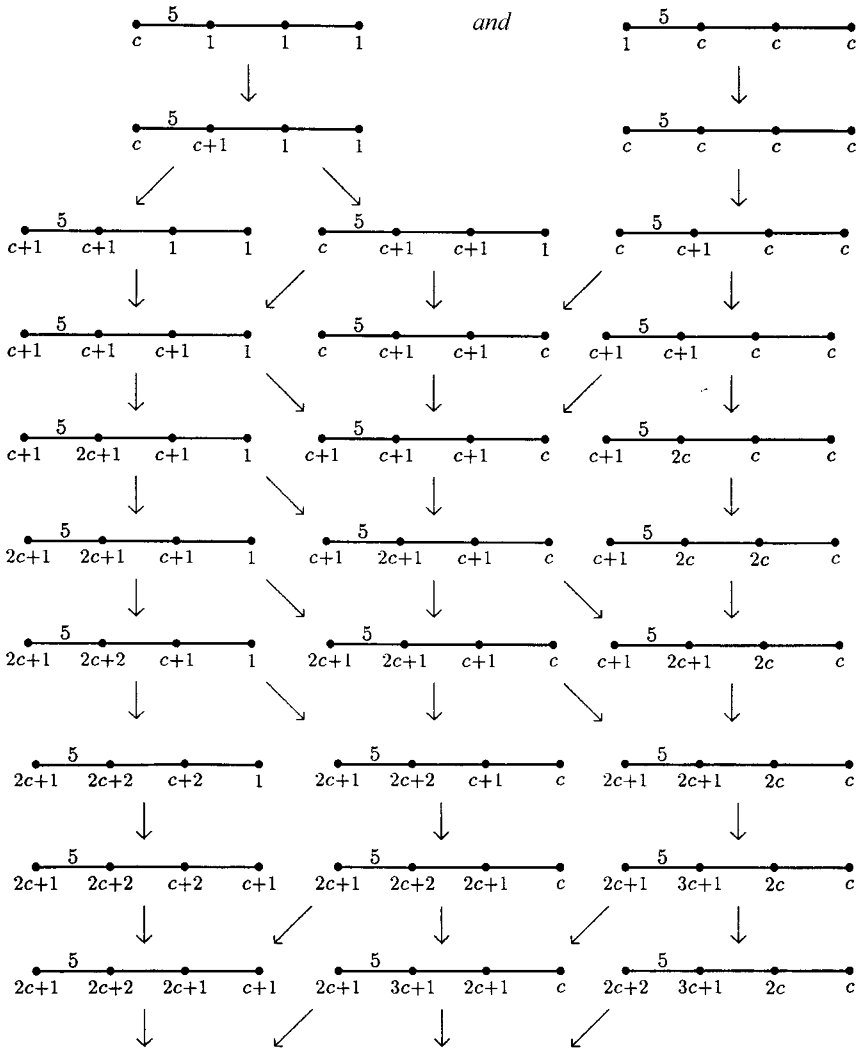
(6.12) PROPOSITION. *Let* $S \subseteq \Pi$ *be such that* $\Gamma(S)$ *is of shape* (1) *with* $m = 5$. *If* $l = n = 1$, *the roots in* $\mathscr{E}_S$ *are*

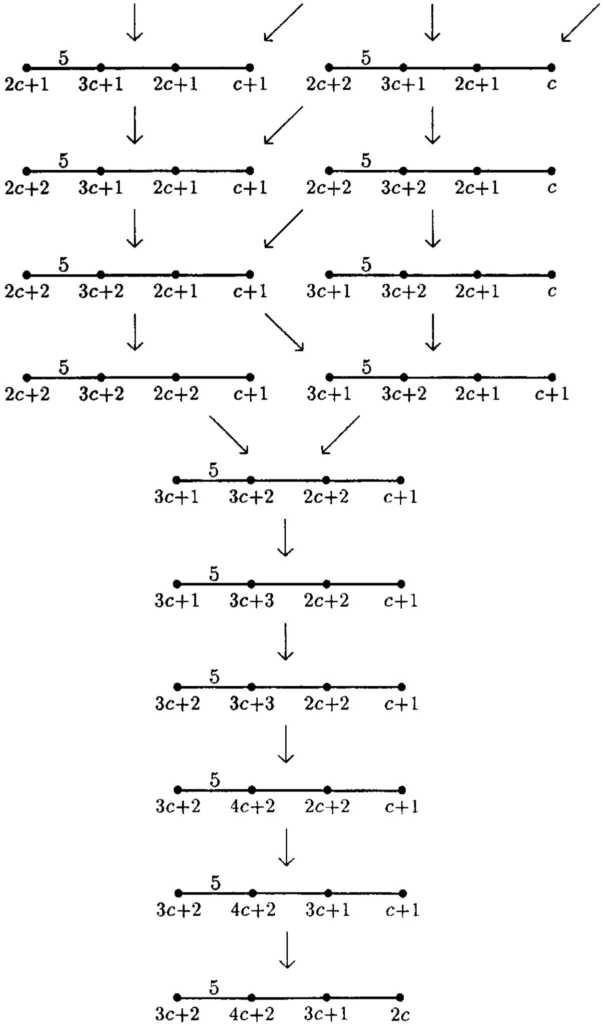


If $l = 1$ *and* $n = 2$, *the roots in* $\mathscr{E}_S$ *are*



If $l = 1$ and $n = 3$, the roots in $\mathscr{E}_S$ are




5
2c+1 3c+1 2c+1 c+1
5
2c+2 3c+1 2c+1 c
5
2c+2 3c+1 2c+1 c+1
5
2c+2 3c+2 2c+1 c
5
2c+2 3c+2 2c+1 c+1
5
3c+1 3c+2 2c+1 c
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2c+2 3c+2 2c+2 c+1
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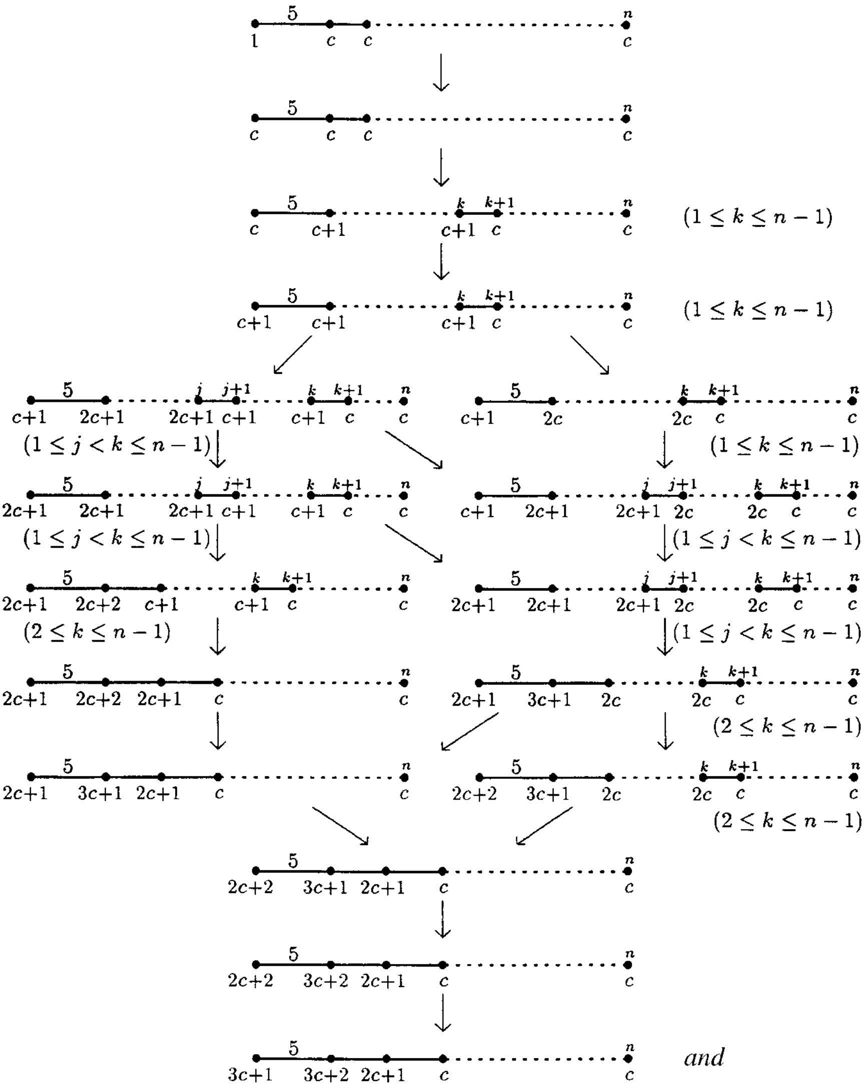
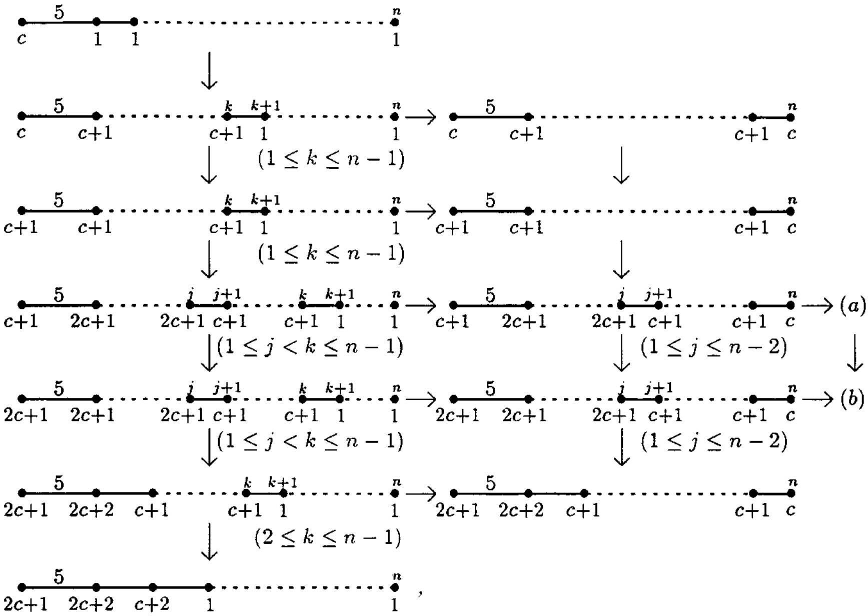
5
3c+2 4c+2 3c+1 2c

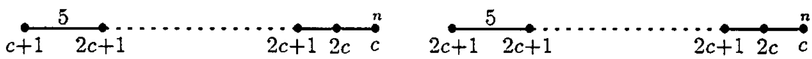
If $l = 1$ and $n \geq 4$, the roots in $\mathscr{E}_S$ are



and



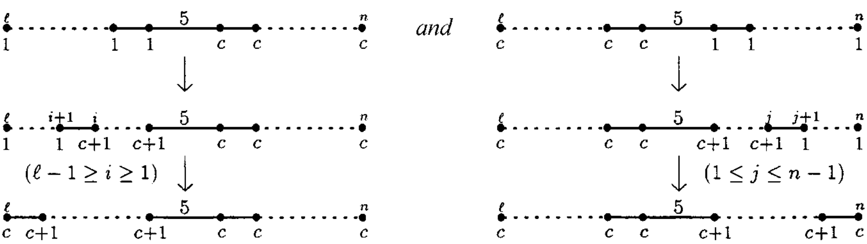
where (a), (b) *are the roots*



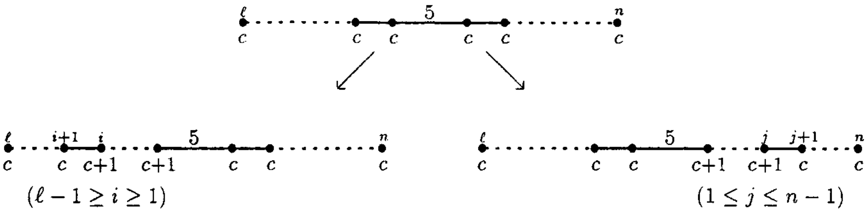
respectively.

($Note$ *that the roots preceded by both* $\alpha_{1,n}$ *and* $\beta_{1,n}$ *are listed twice.*)

If l, $n \geq 2$, *the roots in* $\mathscr{E}_S$ *are*



as well as



(*Note that the roots preceded by more than one of* $\alpha_{l,n}, \beta_{l,n}, \gamma_{l,n}$ *are listed twice*.)

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